On de Rham–Witt Cohomology of Classifying Stacks

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Abstract

We give an example of proper smooth fourfold over a perfect field k of characteristic $p > 0$ with asymmetric Hodge–Witt numbers in total degree 3. Our example is sharp both in terms of dimension and total degree. We arrive at our example by computing and approximating the Hodge–Witt cohomology groups of the classifying stack $B\alpha_p$.

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1 Introduction

de Rham–Witt complex of a smooth variety X over a perfect field k of positive characteristic was introduced by Illusie [Ill79], and further studied by Illusie–Raynaud [IR83], Ekedahl [Eke84, Eke85] and others in the 1980s. It plays a central role in the study of the arithmetic of such varieties.

When X is further assumed to be proper, associated with the cohomology of its de Rham–Witt complex is a collection of numerical invariants: Namely the *Hodge–Witt numbers* $h_W^{i,j}(X)$ introduced by Ekedahl [Eke86, Definition IV.3.1]. These numbers arise from the homological algebra of the de Rham–Witt complex and are defined in terms of slope multiplicities and so-called "*domino numbers*". Although their definition is somewhat indirect, Crew's formula [Cre85, Theorem 4] and Ekedahl's inequality theorem [Eke86, Theorem IV.3.3] show that they are closely related to the classical Hodge numbers $h^{i,j}(X)$. From this perspective, the Hodge–Witt numbers may be regarded as characteristic p analogues of Hodge numbers. The search for such an analogue is reasonable, as it is well-known that Hodge numbers in positive characteristics behave very poorly.

It is natural to ask to what extent the familiar symmetries of Hodge numbers persist for Hodge–Witt numbers. Ekedahl proved that they satisfy the analogue of Serre duality [Eke86, Proposition VI.3.2]:

$$h_W^{i,j} = h_W^{N-i,N-j},$$

whenever X is proper and smooth of equidimension N. This symmetry reflects the duality theory of Hodge–Witt cohomology. Another fundamental symmetry of Hodge numbers in characteristic 0 is *Hodge symmetry*,

$$h^{i,j} = h^{j,i}.$$

While knowing that in positive characteristic Hodge numbers can be asymmetric, it is natural to ask whether the same symmetry holds for Hodge–Witt numbers:

$$h_W^{i,j} = h_W^{j,i}.$$

In low degree or dimension, such symmetry holds true: For instance, Ekedahl proved that Hodge symmetry holds for Hodge–Witt numbers whenever $i+j \leqslant 2$ [Eke86, Corollary VI.3.3(ii)], and more generally for all smooth proper varieties of dimension $\leqslant 3$ [Eke86, Corollary VI.3.3(iii)]. Beyond these cases, however, no general principle is known that would force "Hodge–Witt symmetry" in higher dimensions. To this question, Ekedahl himself commented [Eke86, Remark in Page. 113] that "he saw no reason why such a symmetry should hold in general". Nevertheless, no counterexample seems to have appeared in the literature. The purpose of this paper is to show that such counterexamples exist starting exactly in dimension 4 and cohomological degree 3:

Theorem 1.1 (Corollary 5.12)**.** *There exists a smooth proper* 4*-fold over any perfect field of characteristic* $p > 0$*, with* $h_W^{0,3} = 1$*,* $h_W^{1,2} = -2$*,* $h_W^{2,1} = 1$*, and* $h_W^{3,0} = 0$*.*

We note that, given Ekedahl's result, our theorem is sharp both in dimension and in total degree. Our approach is inspired by the work of Antieau–Bhatt–Mathew [ABM21].

Below let us describe the contents of each section. In Section 4, we define de Rham–Witt cohomology of smooth geometric Artin stacks and establish several of its various basic properties. A key result (Theorem 4.9) shows that if the stack is Hodge-proper in the sense of Kubrak–Prikhodko [KP22, Definition 0.2.1], then its de Rham–Witt cohomology groups are coherent in the sense of Illusie–Raynaud [IR83, Définition I.3.8].

In Section 5, we analyze the stack $B\alpha_p$ and compute its low-degree Hodge–Witt cohomologies. These computations show that the Hodge–Witt numbers of $B\alpha_p$ fail to satisfy the analogue of Hodge symmetry for $i+j=3$. Following the strategy of [ABM21], we then approximate the stack $B\alpha_p \times B\mathbb{G}_m$ by smooth proper varieties, thereby producing the counterexample stated in Theorem 1.1.

The computations for the stack $B\alpha_p$ do not come from nowhere; rather, they are built upon fundamental work of Illusie, Illusie–Raynaud, and Ekedahl. In Section 2, we review the theory of graded left $\mathcal{R}$-modules developed by Illusie–Raynaud and Ekedahl, together with their contributions to de Rham–Witt cohomology groups. The Section 3 reviews and extends Ekedahl's results on the deep structures of the derived ∞-category $\mathcal{DG}r(\mathcal{R})$ of graded left $\mathcal{R}$-complexes. Along the way, we reinterpret parts of the theory using the modern language of ∞-categories: We try our best to extend Ekedahl's construction from the level of homotopy categories to the level of ∞-categories.

Notation and conventions

Throughout the article, we shall denote by k a perfect field of characteristic $p > 0$. We use W to denote the ring of Witt vectors of k. We shall use σ to denote the Frobenius operator on W: for any element $a \in W$, we write both a^σ and $\sigma(a)$ for its image under Frobenius.

We write $\mathcal{R}$ for the *Cartier–Dieudonné–Raynaud* graded ring, see Definition 2.1. This is an ordinary associative graded $\mathbb{Z}_p$-algebra. We denote by $\mathcal{DG}r(\mathcal{R})$ the ∞-category $\mathrm{LMod}_{\mathcal{R}}(\mathcal{DG}r(\mathbb{Z}_p))$ of left $\mathcal{R}$-module objects in the graded derived ∞-category of $\mathbb{Z}_p$. Whenever we refer to a graded left $\mathcal{R}$-module or left graded R-module, we mean an object in the heart of the standard t-structure on $\mathcal{DG}r(\mathcal{R})$.

Let $M \in \mathcal{DG}r(\mathcal{R})$. Then each cohomology group $\mathrm{H}^j(M)$ is a graded $\mathcal{R}$-module, and each graded component M^i can be viewed as an object in $\mathcal{D}(W)$. Following Ekedahl's convention, we say that M is *bounded below* (resp. *bounded above*) if $\mathrm{H}^j(M) = 0$ for $j \ll 0$ (resp. $j \gg 0$). Similarly, we say M is *grading-left bounded* (resp. *grading-right bounded*) if $M^i = 0$ for $i \ll 0$ (resp. $i \gg 0$). The grading i piece of $\mathrm{H}^j(M)$ is denoted by $\mathrm{H}^j(M)^i$.

Following [Eke84, Section 0, p. 188], we denote cohomological and grading shifts of M as follows. For integers (i, j), we write $M(i)[j]$ for the object obtained from M by shifting i degrees in the $\mathcal{R}$-grading (horizontally) and j degrees in the cohomological grading (vertically). As a result, there are canonical isomorphisms $\mathrm{H}^n(M(i)[j])^m \cong \mathrm{H}^{n+j}(M)^{(m+i)}$. In contrast, Illusie–Raynaud [IR83] denotes the $\mathcal{R}$-module grading shifting by $M[i]$, which we do not follow because it conflicts with the notation of the cohomological shift.

Occasionally we shall encounter other ordinary graded rings, such as $W[d]/d^2$ or $k[d]/d^2$, we shall use similar notation such as $\mathcal{DG}r(W[d]/d^2)$ and $\mathcal{DG}r(k[d]/d^2)$ and so on.

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2 Left graded $\mathcal{R}$-modules

It was discovered by Illusie that the crystalline complex admits a canonical representative, namely the de Rham–Witt complex. Its existence is rather striking: it provides a concrete complex equipped with Frobenius F and Verschiebung V satisfying the fundamental relation $FdV = d$, thereby encoding the structure of crystalline cohomology. Illusie also observed the appearance of certain torsion phenomena in the slope spectral sequence in the study of supersingular abelian and K3 surfaces ([Ill79, Example II.7.1, II.7.2]). It soon became clear that the failure of degeneration for the slope spectral sequence is largely governed by certain objects now known as dominoes (see Definition 2.15). Illusie and Raynaud formalized this observation by introducing the notion of coherent graded $\mathcal{R}$-modules. The homological algebra of such

$\mathcal{R}$-modules reveals a remarkably rich structure and gives rise to a number of subtle numerical invariants associated with de Rham–Witt cohomology of smooth proper varieties over k.

2.1 The ring $\mathcal{R}$ and totalization

In the 1970s, Illusie introduced and studied in [Ill79] the de Rham–Witt complex $W\Omega^\bullet_{-/k}$ as a contravariant functor defined on all smooth schemes over k (see also [BLM21]). This is a complex of étale sheaves equipped with two operators: Frobenius F and Verschiebung V, satisfying the relations

$$FV = VF = p, \qquad Fa = a^\sigma F, \qquad Va = a^{\sigma^{-1}} V, \qquad FdV = d.$$

To analyze the structure of the cohomology of these de Rham–Witt complexes, Illusie and Raynaud [IR83] introduced a certain graded ring $\mathcal{R}$:

Definition 2.1. Let k be a perfect field of characteristic $p > 0$. The *Cartier–Dieudonné–Raynaud ring* over k is the $\mathbb{Z}$-graded ring $\mathcal{R} = \mathcal{R}^0 \oplus \mathcal{R}^1$, where $\mathcal{R}^0 \cong W_\sigma[F, V]$ is the (noncommutative) Dieudonné ring generated by Frobenius F and Verschiebung V, subject to the relations

$$Fa = a^\sigma F, \qquad Va = a^{\sigma^{-1}} V, \qquad FV = VF = p.$$

The grading 1 component $\mathcal{R}^1$ is the two-sided $\mathcal{R}^0$-module generated by an element d, satisfying

$$d^2 = 0, \qquad FdV = d.$$

Concretely, the graded components of $\mathcal{R}$ can be described as:

$$\begin{aligned} \mathcal{R}^0 &\cong \left\{ \sum_{n>0} a_{-n} F^n + a_0 + \sum_{n>0} a_n V^n \;\middle|\; a_n \in W \text{ and is 0 for all but finitely many } n \right\}, \\ \mathcal{R}^1 &\cong \left\{ \sum_{n>0} a_{-n} F^n d + a_0 d + \sum_{n>0} a_n dV^n \;\middle|\; a_n \in W \text{ and is 0 for all but finitely many } n \right\}. \end{aligned} \tag{2.2}$$

Given a complex

$$\cdots \longrightarrow M^i \longrightarrow M^{i+1} \longrightarrow \cdots$$

in which each M^i is a left $\mathcal{R}^0$-module and the compatibility relation $FdV = d$ holds, the direct sum totalization $\bigoplus_{i\in\mathbb{Z}} M^i$ naturally acquires the structure of a left graded $\mathcal{R}$-module. Conversely, any left graded $\mathcal{R}$-module determines such a compatible complex. In particular, the de Rham–Witt complex $W\Omega^\bullet_{X/k}$ can be regarded as a sheaf of left graded $\mathcal{R}$-modules.

Following [IR83], let $\mathcal{DG}r(\mathcal{R})$ denote the ∞-category of left $\mathcal{R}$-module objects in the graded derived ∞-category $\mathcal{DG}r(\mathbb{Z}_p)$. The forgetful functor induced by the graded ring map $W[d]/d^2 \to \mathcal{R}$ gives rise to a functor $\mathcal{DG}r(\mathcal{R}) \longrightarrow \mathcal{DG}r(W[d]/d^2)$. The ∞-category $\mathcal{DG}r(W[d]/d^2)$ admits a more familiar description, which we briefly recall below.

Digression 2.3 ([Ari21, Ant24])**.** Recall Joyal's pointed 1-category Ξ, whose set of objects is $\mathbb{Z} \cup \{*\}$, where $*$ is both initial and final, and the morphism sets between integers are given by

$$\mathrm{Hom}_{\Xi}(m,n) = \begin{cases} \{*\} & \text{if } n \neq m,\ m-1, \\ \{*, \mathrm{id}\} & \text{if } n = m, \\ \{*, \partial\} & \text{if } n = m-1. \end{cases}$$

Let $\mathcal{C}$ be a small stable ∞-category admitting sequential limits. Denote by $\mathrm{Ch}^{\bullet}(\mathcal{C})$ the ∞-category of pointed functors from Ξ^{op} to $\mathcal{C}$. A result of Ariotta [Ari21, Theorem 4.7] (see also [Ant24, Theorem 3.21]) establishes an equivalence of ∞-categories $\mathrm{Ch}^{\bullet}(\mathcal{C}) \simeq \widehat{\mathcal{DF}}(\mathcal{C})$, where the right-hand side denotes the ∞-category of completely filtered objects in $\mathcal{C}$. Moreover, by [Ari21, Remark 4.9], this equivalence intertwines the functor $(\text{-})\Big|_{\{n\}} : \mathrm{Ch}^{\bullet}(\mathcal{C}) \longrightarrow \mathcal{C}$ and $\mathrm{gr}^n[n] : \widehat{\mathcal{DF}}(\mathcal{C}) \longrightarrow \mathcal{C}$.

Construction 2.4. Let A be a commutative ring and consider the graded associative ring $A[d]/d^2$, where d is placed in grading 1. We specialize the discussion of Digression 2.3 to the case $\mathcal{C} = \mathcal{D}(A)$. There is a natural pointed functor $A[d]/d^2(\text{-})\colon\ \Xi \to \mathcal{DG}r(A[d]/d^2)$ which sends $n \in \Xi$ to $A[d]/d^2(-n)$ and the morphism $n \xrightarrow{\partial} n-1$ to the map of right multiplication by d: $A[d]/d^2(-n) \to A[d]/d^2(1-n)$. Here, for a graded object $M^{\bullet}$, we adopt the convention that the twist is defined by $M(n)^i \coloneqq M^{i+n}$. [1]

Proposition 2.5. *Let the notation be as in Construction 2.4. Then the composite of the Yoneda embedding with restriction along the (opposite of the) functor $A[d]/d^2(\bullet)$ constructed above*

$$\mathcal{DG}r(A[d]/d^2) \xrightarrow{\text{Yoneda}} \mathrm{Fun}_*\big(\mathcal{DG}r(A[d]/d^2)^{\mathrm{op}}, \mathcal{D}(A)\big) \xrightarrow{|_{\Xi^{\mathrm{op}}}} \mathrm{Ch}^{\bullet}(\mathcal{D}(A))$$

is an equivalence of ∞-categories.

Denote this composite functor by $\underline{(-)}\colon \mathcal{DG}r(A[d]/d^2) \longrightarrow \mathrm{Ch}^{\bullet}(\mathcal{D}(A))$.

Proof. Concretely, the composite sends an object $M \in \mathcal{DG}r(A[d]/d^2)$ to the functor $\underline{M}$ with $\underline{M}(n) \coloneqq R\mathrm{Hom}_{\mathcal{DG}r(A[d]/d^2)}\big(A[d]/d^2(n), M\big) \in \mathcal{D}(A)$. It is formal that the functor $\underline{(-)}$ preserves limits. Since $A[d]/d^2(n)$ is compact for every integer n, the functor $\underline{(-)}$ also preserves colimits. Moreover, the canonical t-structures on both sides are left and right t-complete, and the functor $\underline{(-)}$ is t-exact.

We now show that $\underline{(-)}$ is an equivalence. For fully faithfulness, since the cohomological and grading shifts of $A[d]/d^2$ generate $\mathcal{DG}r(A[d]/d^2)$ under colimits, it suffices to prove that the natural map $\mathrm{RHom}_{\mathcal{DG}r(A[d]/d^2)}\big(A[d]/d^2(m)[n], M\big) \longrightarrow \mathrm{RHom}_{\mathrm{Ch}^{\bullet}(\mathcal{D}(A))}\big(\underline{A[d]/d^2(m)[n]}, \underline{M}\big)$ is an equivalence for all integers m, n and all M. Since $\underline{(-)}$ preserves limits, using the Postnikov filtration of M, we are furthur reduced to the case where M is a shift of an ordinary graded $A[d]/d^2$-module. Because $\underline{(-)}$ commutes with shifts, we may further reduce to the case $n = 0$ and M concentrated in cohomological degree 0. In this situation both sides identify with the grading $-m$ component of M, and the induced map is the identity. As for essential surjectivity, since

- the functor $\underline{(-)}$ preserves both limits and colimits,
- both categories are left and right t-complete, and
- the functor $\underline{(-)}$ is t-exact,

[1] As a sanity check, $A[d]/d^2(1)$ is supported in gradings -1 and 0, and multiplication by d indeed defines a map $A[d]/d^2 \to A[d]/d^2(1)$.

it suffices to show that $\underline{(-)}$ induces an equivalence on the hearts: We may check explicitly that both hearts are given by cochain complexes in A-modules and the induced functor is identity. $\square$

Remark 2.6. The equivalence $\underline{(-)}$ intertwines the functors $(-)^n : \mathcal{DG}r(A[d]/d^2) \longrightarrow \mathcal{D}(A)$ and $(-)\Big|_{\{n\}}$: $\mathrm{Ch}^\bullet(\mathcal{D}(A)) \longrightarrow \mathcal{D}(A)$. Indeed, for any $M \in \mathcal{DG}r(A[d]/d^2)$ we have

$$\mathrm{RHom}_{\mathcal{DG}r(A[d]/d^2)}\big(A[d]/d^2(-n), M\big) \cong \mathrm{RHom}_{\mathcal{DG}r(A)}\big(A(-n), M\big) \cong M^n.$$

Here the first equivalence follows from the tensor–Hom adjunction, and the second is tautological from the definition of the grading.

Combining Digression 2.3, Proposition 2.5, and Remark 2.6, we obtain the following.

Corollary 2.7. *Let the notation be as in Construction 2.4. Then there is an equivalence* $\mathcal{DG}r(A[d]/d^2) \longrightarrow \widehat{\mathcal{DF}}(A)$. *Moreover, this equivalence intertwines the functors* $(-)^n : \mathcal{DG}r(A[d]/d^2) \longrightarrow \mathcal{D}(A)$ *and* $\mathrm{gr}^n[n] : \widehat{\mathcal{DF}}(A) \longrightarrow \mathcal{D}(A)$.

Notation 2.8. We denote the functor $\mathcal{DG}r(A[d]/d^2) \longrightarrow \widehat{\mathcal{DF}}(A)$ by $\mathrm{Fil}^\bullet\mathrm{Tot}$, referring to it as the *filtered totalization*. We further denote by $\mathrm{Tot}\colon \mathcal{DG}r(A[d]/d^2) \longrightarrow \mathcal{D}(A)$ the composite of this functor with the forgetful functor sending a filtered object to its underlying object; we refer to this as the *totalization*.

Remark 2.9. Unwinding the construction, one finds that if a cochain complex of A-modules is viewed as a graded $A[d]/d^2$-module, then its filtered totalization is canonically identified with the same cochain complex equipped with the *stupid filtration*.

In view of the totalization functor defined above, the result of Illusie [Ill79, Theorem II.1.4] can be re-stated as follows (see also [BLM21, Theorem 1.1.2]):

Theorem 2.10. *There is a commutative diagram:*

$$\begin{array}{ccc} \mathrm{SmSch}_k^{\mathrm{op}} & \xrightarrow{\mathrm{R\Gamma}(-,W\Omega^\bullet)} & \mathcal{DG}r(\mathcal{R}) \\ & {\scriptstyle \mathrm{R\Gamma_{crys}}(-/W(k))} \searrow \quad \swarrow {\scriptstyle \mathrm{Tot}\circ\mathrm{forget}} & \\ & \mathcal{D}(W(k)). & \end{array}$$

Proof. By [BLM21, Remark 9.3.5] together with Remark 2.9, there is a canonical natural transformation

$$\mathrm{R\Gamma}(-, W\Omega^\bullet) \longrightarrow \mathrm{R\Gamma_{crys}}(-/W(k))$$

when both functors are regarded as defined on affine smooth k-schemes. By [Ill79, Theorem II.1.4], this natural transformation is an equivalence.

Note that although the functor Tot does not commute with limits in general, it does commute with limits when the objects involved are uniformly grading-left bounded; that is, when there exists an integer N such that all grading pieces $\leqslant -N$ vanish. The reason being, in this situation, the totalization is the same as taking the $(-N)$-th filtration piece, which clearly commutes with arbitrary limits.

The general case follows from the fact that both functors are Zariski sheaves. $\square$

For later use we record the following observation.

Proposition 2.11. *Let the notation be as in Construction 2.4. Then the composite*

$$\mathcal{DG}r(A) \xrightarrow{A[d]/d^2\otimes_A -} \mathcal{DG}r(A[d]/d^2) \xrightarrow{\mathrm{Tot}} \mathcal{D}(A)$$

is naturally equivalent to the zero functor.

Proof. For an integer j, instead of taking the underlying object of the filtered totalization, consider the "totalization modulo the $(j+1)$-st filtration", which we denote by $\mathrm{Tot}/\mathrm{Fil}^{j+1}\colon \mathcal{DG}r(A[d]/d^2) \to \mathcal{D}(A)$. This functor commutes with small colimits. Hence the composite $\mathcal{DG}r(A) \xrightarrow{A[d]/d^2\otimes_A -} \mathcal{DG}r(A[d]/d^2) \xrightarrow{\mathrm{Tot}/\mathrm{Fil}^{j+1}} \mathcal{D}(A)$ also commutes with small colimits. We claim that this composite is naturally equivalent to the functor $(-)^j[-j]\colon \mathcal{DG}r(A) \longrightarrow \mathcal{D}(A)$.

First, let us construct a natural transformation. Consider the natural transformation of functors

$$\mathrm{gr}^j \longrightarrow \mathrm{Tot}/\mathrm{Fil}^{j+1}\colon \mathcal{DG}r(A[d]/d^2) \longrightarrow \mathcal{D}(A).$$

By the second part of Corollary 2.7, we compute

$$\mathrm{gr}^j \circ (A[d]/d^2 \otimes_A -) \cong (A[d]/d^2 \otimes_A -)^j[-j] \cong \big((-)^{j-1} \oplus (-)^j\big)[-j]\colon \mathcal{DG}r(A) \longrightarrow \mathcal{D}(A).$$

Precomposing with the inclusion of the direct summand $(-)^j \hookrightarrow (-)^{j-1} \oplus (-)^j$ produces the desired natural transformation. We next show that this natural transformation is an equivalence. Since both functors preserve colimits and $\mathcal{DG}r(A)$ is generated under colimits by the objects $A(m)[n]$ for all integers m, n, it suffices to verify the claim on these generators. For such objects the statement follows by direct inspection.

Finally, we compute the limit $\mathrm{Rlim}_{j\to\infty}\mathrm{Tot}/\mathrm{Fil}^{j+1}\circ(A[d]/d^2\otimes_A -) \cong \mathrm{Rlim}_{j\to\infty}(-)^j[-j]$. We do not need to identify the transition maps explicitly. Indeed, we claim that any natural transformation $(-)^i \to (-)^j[n]$ vanishes whenever $i \neq j$. This immediately implies that the above limit is zero. To prove the claim, observe that every object of $\mathcal{DG}r(A)$ admits an action of the ring $\prod_{\mathbb{Z}} A$ by grading projections. Let e_i denote the idempotent projecting to the i-th grading component. Then the endomorphism $1 - e_i$ acts as 0 under the functor $(-)^i$, but as the identity under $(-)^j[n]$ when $i \neq j$. Naturality therefore forces any such transformation to vanish. □

2.2 Coherent $\mathcal{R}$-modules

As a consequence of Theorem 2.10, one obtains the so-called "slope spectral sequence"

$$E_1^{i,j} \coloneqq \mathrm{H}^j(X, W\Omega^i_{X/k}) \Longrightarrow \mathrm{H}^{i+j}_{\mathrm{crys}}(X/W). \tag{2.12}$$

The main structural result of [Ill79] may be stated as follows.

Theorem 2.13 ([Ill79, Theorem II.3.2])**.** *For a smooth proper variety X over k. Its associated slope spectral sequence (2.12) degenerates at the E_1-page after inverting p. Moreover, the isocrystal $\mathrm{H}^j(X, W\Omega^i_{X/k})\otimes_W K$, endowed with Frobenius p^iF, coincides with the slope interval $[i, i+1)$ of $\mathrm{H}^{i+j}_{\mathrm{crys}}(X/W) \otimes_W K$.*

However, without inverting p—that is, when the p^∞-torsion is taken into account—the spectral sequence (2.12) need not degenerate at the E_1-page. In fact, it is known that the non-degeneration of the slope spectral sequence is equivalent to non-finiteness of Hodge–Witt cohomology groups. In particular, the

Hodge–Witt cohomology groups of smooth proper varieties over k can be non-finitely generated as W-modules. As a replacement for the correct "finiteness" condition for Hodge–Witt cohomology groups of smooth proper varieties over k, Illusie–Raynaud [IR83, Remark I.3.10.1] and Ekedahl [Eke85, Definition 0.5.13] introduced a distinguished subcategory of graded left $\mathcal{R}$-modules consisting of so-called *coherent* graded $\mathcal{R}$-modules (see also [Eke85, Proposition III.1.1]). We briefly recall their definition.

Definition 2.14. Let M be a left graded $\mathcal{R}$-module. The *standard filtration* on M is defined by

$$\mathrm{Fil}^n(M^i) := V^n M^i + dV^n M^{i-1}.$$

These form graded W-submodules of M. We say that M is *classically complete* if for each i the natural map $M^i \to \varprojlim_n M^i/\mathrm{Fil}^n(M^i)$ is an isomorphism. We say that M is *profinite* if it is classically complete and each quotient $M^i/\mathrm{Fil}^n(M^i)$ has finite length as a W-module.

In the study of Hodge–Witt cohomology, a fundamental class of graded $\mathcal{R}$-modules naturally arises. These are the modules U_t, known as *elementary dominoes*, which first appear in the cohomology of supersingular abelian surfaces and K3 surfaces.

Definition 2.15 ([IR83, I.2.(D)])**.** For each integer $t \in \mathbb{Z}$, the left graded $\mathcal{R}$-module $U_t = U_t^0 \oplus U_t^1$ is defined by:

1. The grading 0 piece is $U_t^0 := k_\sigma[[V]] = \left\{\sum_{n\geqslant 0} a_n V^n \mid a_n \in k\right\}$, where F acts by 0 and
$$V\left(\sum_{n\geqslant 0} a_n V^n\right) = \sum_{n\geqslant 0} a_n^{\sigma^{-1}} V^{n+1};$$
2. The grading 1 piece is $U_t^1 := \prod_{n\geqslant t} k\cdot dV^n$, where V acts by 0 and
$$F\left(\sum_{n\geqslant t} a_n dV^n\right) = \sum_{n\geqslant t} a_{n+1}^{\sigma} dV^n.$$
In the above definition, we adopt the convention that $dV^n = F^{-n}d$ when $n \leqslant 0$.
3. The action of d is given by
$$d\left(\sum_{n\geqslant 0} a_n V^n\right) = \sum_{n\geqslant \max\{0,t\}} a_n dV^n.$$

For $t \geq 0$, the structure of U_t may be illustrated as follows (see also [IR83, p. 108]):

$$\begin{array}{lccccccccccccc}
U_t^0 : & k & \xrightarrow{V} & kV & \xrightarrow{V} & \cdots & \xrightarrow{V} & kV^{i-1} & \xrightarrow{V} & kV^i & \xrightarrow{V} & kV^{i+1} & \xrightarrow{V} & \cdots \\
{\scriptstyle d}\big\downarrow & & & & & & & & & {\scriptstyle d}\big\downarrow & & {\scriptstyle d}\big\downarrow & & \\
U_t^1 : & & & & & & & 0 & \xleftarrow{F} & kdV^i & \xleftarrow{F} & kdV^{i+1} & \xleftarrow{F} & \cdots
\end{array}$$

Illusie and Raynaud [IR83, Proposition I.2.19] proved that every grading-left bounded (resp. grading-left and right bounded) profinite left graded $\mathcal{R}$-module admits a separated and exhaustive (resp. finite) increasing filtration by graded $\mathcal{R}$-submodules whose graded pieces are, up to shift, of one of the following types:

Type I_a: a finite length Dieudonné module, with V nilpotent;

Type I_b : a finite free Dieudonné module, with V topologically nilpotent[2];

Type I_c: the module $k_\sigma[[V]]$, defined as in Definition 2.15;

Type II: an elementary domino U_t also defined in Definition 2.15.

Definition 2.16. Let M be a profinite left graded $\mathcal{R}$-module.

1. ([IR83, Définition I.3.8]) We say that M is *coherent* if it admits a finite filtration by graded $\mathcal{R}$-submodules whose successive quotients are of type I_a, I_b, or type II. In particular, coherent left graded $\mathcal{R}$-modules have bounded gradings.
2. ([IR83, Définition II.3.1]) The *heart* of M^i is defined by

$$\text{cœur}(M^i) := V^{-\infty}Z^i/F^\infty B^i,$$

where

$$V^{-\infty}Z^i := \bigcap_n \ker(dV^n : M^i \to M^{i+1}), \qquad F^\infty B^i := \bigcup_n F^n \operatorname{im}(d : M^{i-1} \to M^i).$$

One checks that $V^{-\infty}Z^i$ is the largest $\mathcal{R}^0$-submodule contained in $\ker(d : M^i \to M^{i+1})$, while $F^\infty B^i$ is the smallest $\mathcal{R}^0$-submodule containing $\operatorname{im}(d : M^{i-1} \to M^i)$.
3. The i-th *domino* of M is the graded $\mathcal{R}$-module

$$M^i/V^{-\infty}Z^i \longrightarrow F^\infty B^{i+1},$$

concentrated in degrees 0 and 1. According to [IR83, Proposition I.2.18], it is an iterated extension of elementary dominoes U_t. The number of such factors is called the i-th domino number of M and is denoted $T^i(M)$ (see the remark below for why this is well-defined).

Remark 2.17. Let M be a profinite left graded $\mathcal{R}$-module concentrated in finitely many gradings. By [IR83, Theorem I.3.8], we have that M is coherent if and only if $\text{cœur}(M^i)$ is finitely generated for all $i \in \mathbb{Z}$. Moreover, by [IR83, Proposition I.2.18], the i-th domino number admits an alternative description

$$T^i(M) := \dim_k M^i/(V^{-\infty}Z^i + VM^i).$$

Assuming that M is coherent, it follows from Corollary 2.41 that $T^i(M)$ is also equal to the number of factors of the form $U_t(-i)$, for varying t, appearing in the graded pieces of any filtration of M satisfying the conclusion of [IR83, Proposition I.2.19].

The category of coherent $\mathcal{R}$ modules is an abelian subcategory [Eke85, page. 55].[3]

Definition 2.18. The category $\mathcal{DG}r^b_c(\mathcal{R})$ (resp. $\mathcal{DG}r^-_c(\mathcal{R})$, resp. $\mathcal{DG}r_c(\mathcal{R})$) is the full subcategory of $\mathcal{DG}r^b(\mathcal{R})$ (resp. $\mathcal{DG}r^-(\mathcal{R})$, resp. $\mathcal{DG}r(\mathcal{R})$) spanned by those objects whose cohomology are all coherent as left graded $\mathcal{R}$-modules.

We now recall several invariants associated with a coherent left graded $\mathcal{R}$-module M, as introduced by Illusie–Raynaud [IR83] and further studied by Crew [Cre85] and Ekedahl [Eke86].

[2]Here the finite free W-module is equipped with the p-adic topology, so concretely we are saying that the operator V^N is divisible by p for some N.

[3]This is a consequence of the fact that $\mathcal{DG}r^b_c(\mathcal{R})$ is a triangulated subcategory, see the discussion after Proposition 2.32.

Definition 2.19. Let $M \in \mathcal{DG}r_c(\mathcal{R})$, following Ekedahl [Eke86, Chapter IV.3.1], we define some numerical invariants:

1. the *domino number* $T^{i,j}(M)$ is defined to be the i-th domino number of the coherent $\mathcal{R}$-module $\mathrm{H}^j(M)$.
2. The *slope number*[4] $m^{i,j}(M)$ is defined by:

$$\begin{aligned} m^{i,j}(M) \coloneqq \sum_{\lambda\in[0,1)} (1-\lambda) m_\lambda(\text{cœur}(\mathrm{H}^j(M)^i)\otimes K) \\ + \sum_{\lambda\in[0,1)} \lambda m_\lambda(\text{cœur}(\mathrm{H}^{j+1}(M)^{i-1})\otimes K). \end{aligned} \tag{2.20}$$

   where $m_\lambda(H)$ denotes the multiplicity of slope λ in an (iso)crystal H.
3. The *Hodge–Witt number* $h_W^{i,j}(M)$ is defined by:

$$h_W^{i,j}(M) \coloneqq m^{i,j}(M) + T^{i,j}(M) - 2T^{i-1,j+1}(M) + T^{i-2,j+2}(M).$$

Remark 2.21. The slope numbers admit a combinatorial interpretation (see [Eke86, Proposition IV.2.5]). Let $M \in \mathcal{DG}r_c(\mathcal{R})$. Then the element $d \in \mathcal{R}^1$ acts by 0 on the graded K-vector space $\mathrm{H}^i(M)[1/p]$. Let n be an integer such that $\bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i(M)^{n-i}[1/p]$ is finite-dimensional over K. We may then consider the isocrystal $\left(\bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i(M)^{n-i}[1/p], \varphi\right)$, where φ acts as p^iF on $\mathrm{H}^i(M)^{n-i}[1/p]$, and form its Newton polygon. The *Newton–Hodge polygon* is the polygon obtained by assigning slope i with multiplicity $m^{i,n-i}$. It is then the maximal convex polygon lying below the Newton polygon with the same endpoints and with integral slopes.

2.3 Ekedahl's completion functor

The following result, due to Illusie–Raynaud, is fundamental:

Theorem 2.22 ([IR83, Theorem II.2.2])**.** *Let X be a proper smooth variety over the perfect field k of dimension n, then*

$$R\Gamma(X, W\Omega^\bullet_{X/k}) \in \mathcal{DG}r_c^b(\mathcal{R}).$$

That is, the left graded $\mathcal{R}$-module associated to the complex

$$\mathrm{H}^j(X, WO_{X/k}) \xrightarrow{d} \mathrm{H}^j(X, W\Omega^1_{X/k}) \xrightarrow{d} \dots \xrightarrow{d} \mathrm{H}^j(X, W\Omega^n_{X/k})$$

is coherent for all j.

Let us sketch an alternative proof of this fact, due to Ekedahl. To do so, we take a brief detour to introduce his completion functor on $\mathcal{DG}r(\mathcal{R})$ (see [Eke85, page 63]).

Digression 2.23 (Naturally induced symmetric monoidal structures)**.** Let $\mathcal{C}^\otimes$ be a small symmetric monoidal ∞-category. Then its opposite category $\mathcal{C}^{\mathrm{op}}$ inherits a natural symmetric monoidal structure in which the tensor product of objects is unchanged; see [Lur17, Remark 2.4.2.7]. There is also a canonical symmetric monoidal structure on Ind-$\mathcal{C}$, the category of Ind-objects of $\mathcal{C}$, characterized by requiring that

[4] In [Eke86, Definition IV.3.4], the Hodge polygon formed by these numbers is called the Newton–Hodge polygon.

the Yoneda embedding $j\colon \mathcal{C} \to \text{Ind-}\mathcal{C}$ be symmetric monoidal and that the tensor product preserve small filtered colimits separately in each variable; see [Lur17, Corollary 6.3.1.13].

Consequently, the category $\text{Pro-}\mathcal{C} = (\text{Ind-}\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ inherits a natural symmetric monoidal structure for which the canonical Yoneda embedding $j\colon \mathcal{C} \to \text{Pro-}\mathcal{C}$ is symmetric monoidal. Unwinding the definitions, if $\{X_\lambda\}_{\lambda\in\Lambda}$ and $\{Y_\gamma\}_{\gamma\in\Gamma}$ are two pro-systems, then their tensor product is given by

$$\{X_\lambda\}_{\lambda\in\Lambda} \otimes_{\text{Pro-}\mathcal{C}} \{Y_\gamma\}_{\gamma\in\Gamma} = \{X_\lambda \otimes_{\mathcal{C}} Y_\gamma\}_{(\lambda,\gamma)\in\Lambda\times\Gamma}.$$

Finally, suppose that $\mathcal{C}$ admits sufficiently many small limits so that the symmetric monoidal functor $j\colon \mathcal{C} \to \text{Pro-}\mathcal{C}$ admits a right adjoint $\lim\colon \text{Pro-}\mathcal{C} \to \mathcal{C}$. Then this right adjoint lim is naturally a lax symmetric monoidal functor.

After this interlude, we can introduce the completion functor. For each $n \geqslant 1$, define $\mathcal{R}_n \coloneqq \mathcal{R}/(V^n\mathcal{R} + dV^n\mathcal{R})$. Then $\mathcal{R}_n$ naturally carries the structure of a graded $(W_n[d]/d^2, \mathcal{R})$-bimodule. The natural projection induces graded maps $\mathcal{R}_{n+1} \xrightarrow{\pi} \mathcal{R}_n$, while left multiplication by F (resp. V, resp. d) induces graded maps $F\colon \mathcal{R}_n \to \sigma_*\mathcal{R}_{n-1}$, $\quad V\colon \sigma_*\mathcal{R}_{n-1} \to \mathcal{R}_n$, $\quad d\colon \mathcal{R}_n \to \mathcal{R}_n(1)$. Consequently, the pro-system $\{\mathcal{R}_n\}_n$ (with transition maps given by π) forms a graded $(\mathcal{R},\mathcal{R})$-bimodule object in $\text{Pro-Mod}_{\mathbb{Z}_p}$, the abelian category of Pro-$\{\mathbb{Z}_p$ modules$\}$. From now on, we shall view it as a graded $(\mathcal{R},\mathcal{R})$-bimodule object in $\text{Pro-}\mathcal{DG}r(\mathbb{Z}_p)$.

Construction 2.24 (completion functor, cf. [Eke85, page 63])**.** Note that $\mathcal{DG}r(\mathcal{R}) = \mathrm{LMod}_{\mathcal{R}}(\mathcal{DG}r(\mathbb{Z}_p))$. Let us consider the following composite functor

$$\mathcal{DG}r(\mathcal{R}) \xrightarrow{j} \mathrm{LMod}_{\mathcal{R}}(\text{Pro-}\mathcal{DG}r(\mathbb{Z}_p)) \xrightarrow{\{\mathcal{R}_n\}_n \otimes^{\mathrm{L}}_{\mathcal{R}} -} \mathrm{LMod}_{\mathcal{R}}(\text{Pro-}\mathcal{DG}r(\mathbb{Z}_p)) \xrightarrow{\lim} \mathcal{DG}r(\mathcal{R}).$$

This defines an endo-functor on $\mathcal{DG}r(\mathcal{R})$, which we call the *completion functor* and denote by $\widehat{(-)}$. Here we use the symmetric monoidal structure on $\text{Pro-}\mathcal{DG}r(\mathbb{Z}_p)$ induced from that on $\mathcal{DG}r(\mathbb{Z}_p)$, as explained in Digression 2.23. Moreover, the functor lim preserves the left $\mathcal{R}$-module structure because it is lax symmetric monoidal (again by Digression 2.23). We shall denote by $\widehat{\mathcal{DG}r(\mathcal{R})}$ the full subcategory of $\mathcal{DG}r(\mathcal{R})$ spanned by the essential image of $\widehat{(-)}$.

Proposition 2.25 ([IR83, Proposition I.3.2])**.** *For each integer $n \geqslant 1$, the right graded $\mathcal{R}$-module $\mathcal{R}_n$ admits the following resolution:*

$$0 \to \mathcal{R}(-1) \xrightarrow{u_n} \mathcal{R}(-1) \oplus \mathcal{R} \xrightarrow{v_n} \mathcal{R} \to \mathcal{R}_n \to 0, \tag{2.26}$$

where $u_n(x) = (F^n x, -F^n dx)$ and $v_n(x,y) = dV^n x + V^n y$.

Moreover, left multiplication by a scalar $c \in W$ is compatible with this resolution and is described by the following commutative diagram of right graded $\mathcal{R}$-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{R}(-1) & \xrightarrow{u_n} & \mathcal{R}(-1)\oplus\mathcal{R} & \xrightarrow{v_n} & \mathcal{R} & \longrightarrow & \mathcal{R}_n & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle c} & & \Big\downarrow{\scriptstyle (\sigma^n(c),\sigma^n(c))} & & \Big\downarrow{\scriptstyle c\cdot} & & \Big\downarrow{\scriptstyle c\cdot} & & \\
0 & \longrightarrow & \mathcal{R}(-1) & \xrightarrow{u_n} & \mathcal{R}(-1)\oplus\mathcal{R} & \xrightarrow{v_n} & \mathcal{R} & \longrightarrow & \mathcal{R}_n & \longrightarrow & 0
\end{array}$$

Similarly, left multiplication by d is described by the following commutative diagram of right graded

$\mathcal{R}$*-modules:*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{R}(-1) & \xrightarrow{u_n} & \mathcal{R}(-1)\oplus\mathcal{R} & \xrightarrow{v_n} & \mathcal{R} & \longrightarrow & \mathcal{R}_n & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle -d\cdot} & & \Big\downarrow{\scriptstyle w_n} & & \Big\downarrow{\scriptstyle d\cdot} & & \Big\downarrow{\scriptstyle d\cdot} & & \\ 0 & \longrightarrow & \mathcal{R} & \xrightarrow{u_n} & \mathcal{R}\oplus\mathcal{R}(1) & \xrightarrow{v_n} & \mathcal{R}(1) & \longrightarrow & \mathcal{R}_n(1) & \longrightarrow & 0 \end{array}$$

where $w_n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

In particular, the derived functor

$$\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} - \colon \mathcal{DG}r(\mathcal{R}) \longrightarrow \mathcal{DG}r(W_n[d]/d^2)$$

has amplitude contained in $[-2, 0]$.

Letting n *vary, the above resolutions are compatible and fit into the following commutative diagram:*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{R}(-1) & \xrightarrow{u_{n+1}} & \mathcal{R}(-1)\oplus\mathcal{R} & \xrightarrow{v_{n+1}} & \mathcal{R} & \longrightarrow & \mathcal{R}_{n+1} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle (V,\, V)} & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow & & \\ 0 & \longrightarrow & \mathcal{R}(-1) & \xrightarrow{u_n} & \mathcal{R}(-1)\oplus\mathcal{R} & \xrightarrow{v_n} & \mathcal{R} & \longrightarrow & \mathcal{R}_n & \longrightarrow & 0 \end{array}$$

Consequently, these resolutions assemble into a resolution of the pro-system $\{\mathcal{R}_n\}$ *as a right graded* $\mathcal{R}$*-module object in* $\text{Pro-Mod}_{\mathbb{Z}_p}$*:*

$$0 \to \{\mathcal{R}(-1)\}_p \xrightarrow{u} \{\mathcal{R}(-1)\oplus\mathcal{R}\}_V \xrightarrow{v} \{\mathcal{R}\} \to \{\mathcal{R}_n\} \to 0. \tag{2.27}$$

Here the three pro-systems are indexed by the natural numbers, with the terms as above, and with transition maps given by left multiplication by p*,* V*, and* 1*, respectively.*

Notice that there is a natural morphism $j(\mathcal{R}) \longrightarrow \{\mathcal{R}_n\}$ of graded $(\mathcal{R},\mathcal{R})$-bimodule objects in Pro-$\mathcal{DG}r(W)$. Consequently, we obtain a natural transformation

$$\eta\colon \mathrm{id} \longrightarrow \widehat{(-)}$$

between endofunctors of $\mathcal{DG}r(\mathcal{R})$.

Proposition 2.28 (cf. [Eke85, Proposition 2.1])**.** *The functor* $\widehat{(-)}$ *is a localization.*

Proof. We verify the criterion of [Lur09, Proposition 5.2.7.4.(3)] for the natural transformation η. Thus it suffices to show that the two natural transformations

$$\eta \circ \widehat{(-)} \ \text{ and } \ \widehat{(\eta)}\colon \widehat{(-)} \longrightarrow \widehat{\widehat{(-)}}$$

are equivalences. Since the forgetful functor $\mathcal{DG}r(\mathcal{R}) \to \mathcal{DG}r(W)$ is conservative, it suffices to verify this after composing with the forgetful functor. Unwinding the definitions, the double completion functor is given by

$$\lim_{\mathbb{N}} \Big(\{\mathcal{R}_n\} \otimes^{\mathrm{L}}_{\mathcal{R}} \big(\lim_{\mathbb{N}}(\{\mathcal{R}_m\} \otimes^{\mathrm{L}}_{\mathcal{R}} -)\big)\Big) \simeq \lim_{\mathbb{N}\times\mathbb{N}} \big(\{\mathcal{R}_n\} \otimes^{\mathrm{L}}_{\mathcal{R}} \{\mathcal{R}_m\} \otimes^{\mathrm{L}}_{\mathcal{R}} -\big).$$

Here the interchange of limits follows from the fact that each $\mathcal{R}_n$ is perfect as a right $\mathcal{R}$-module (see (2.26)), so that the functor $\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} -$ commutes with limits.

Under this identification, it suffices to show that both maps

$$\mathrm{id}\otimes 1 \text{ and } 1\otimes\mathrm{id}\colon \{\mathcal{R}_n\} \longrightarrow \{\mathcal{R}_n\}\otimes^{\mathrm{L}}_{\mathcal{R}}\{\mathcal{R}_n\}$$

are equivalences in $\mathrm{RMod}_{\mathcal{R}}(\text{Pro-}\mathcal{DG}r(W))$. Applying the resolution (2.27) to the first copy of $\{\mathcal{R}_n\}$, we find that $\{\mathcal{R}_n\}\otimes^{\mathrm{L}}_{\mathcal{R}}\{\mathcal{R}_n\}$ is represented by the following complex of pro-systems, concentrated in cohomological degrees $[-2,0]$:

$$\text{“}\lim_{p\cdot -}\text{”}\{\mathcal{R}_n(-1)\} \longrightarrow \text{“}\lim_{V\cdot -}\text{”}\{\mathcal{R}_n(-1)\oplus\mathcal{R}_n\} \longrightarrow \{\mathcal{R}_n\}.$$

Moreover, the morphism $\mathcal{R}\otimes_{\mathcal{R}}\{\mathcal{R}_n\} \xrightarrow{1\otimes\mathrm{id}} \{\mathcal{R}_n\}\otimes^{\mathrm{L}}_{\mathcal{R}}\{\mathcal{R}_n\}$ is induced by the stupid truncation of this representing complex. Since the first two terms are pro-0[5], it follows that $1\otimes\mathrm{id}$ is an equivalence.

On the other hand, the composite $(1\otimes\mathrm{id})^{-1}\circ(\mathrm{id}\otimes 1)$ is an endomorphism of $\{\mathcal{R}_n\}$ which commutes with the canonical $(\mathcal{R},\mathcal{R})$-bimodule map $j(\mathcal{R})\to\{\mathcal{R}_n\}$. As a graded right $\mathcal{R}$-module morphism, such an endomorphism must necessarily be the identity. Hence $\mathrm{id}\otimes 1$ is also an equivalence, completing the proof. □

Proposition 2.29. *The full subcategory $\widehat{\mathcal{DG}r}(\mathcal{R})\subset\mathcal{DG}r(\mathcal{R})$ is closed under limits.*

Proof. By Proposition 2.28, it suffices to show that the completion functor

$$\widehat{(-)} \coloneqq \mathcal{R}\lim_{n\in\mathbb{N}}\big(\mathcal{R}_n\otimes^{\mathrm{L}}_{\mathcal{R}} -\big)$$

commutes with small limits. This follows from the fact that both $\mathcal{R}\lim_{n\in\mathbb{N}}(-)$ and the functor $\mathcal{R}_n\otimes^{\mathrm{L}}_{\mathcal{R}} -$ commute with small limits. The latter property holds because each $\mathcal{R}_n$ is perfect as a right $\mathcal{R}$-module (see Proposition 2.25). □

Let us also record the following important fact.

Proposition 2.30 ([Eke85, Proposition 2.1])**.** *The natural transformation of functors*

$$\mathcal{R}_n\otimes^{\mathrm{L}}_{\mathcal{R}} - \xrightarrow{\mathcal{R}_n\otimes^{\mathrm{L}}_{\mathcal{R}}\eta} \mathcal{R}_n\otimes^{\mathrm{L}}_{\mathcal{R}}\widehat{(-)}\colon \mathcal{DG}r(\mathcal{R}) \longrightarrow \mathcal{DG}r(W_n[d]/d^2)$$

is an equivalence.

Warning 2.31. We warn the readers that for a left graded $\mathcal{R}$-module M, it is unclear to us if being classically complete as a graded left $\mathcal{R}$-module in Definition 2.14 is related to it being complete when viewed as an object in $\mathcal{DG}r(\mathcal{R})$ in the sense of Construction 2.24.

From now on, unless stated otherwise we always use “complete” in the sense of Construction 2.24.

Proposition 2.32 ([Eke85, Proposition III.1.1])**.** *Let $M\in\mathcal{DG}r^-(\mathcal{R})$, then the following conditions are equivalent:*

(1) *$M\in\mathcal{DG}r^-_c(\mathcal{R})$;*

(2) *M is complete and for all n the object $\mathcal{R}_n\otimes^{\mathrm{L}}_{\mathcal{R}} M\in\mathcal{DG}r^-(W_n)$ obtained by forgetting grading and the action of d has finitely generated cohomology.*

(3) *M is complete and $\mathcal{R}_1\otimes^{\mathrm{L}}_{\mathcal{R}} M\in\mathcal{DG}r^-(k)$ obtained by forgetting grading and the action of d has finite dimensional cohomology.*

[5]Indeed, each $\mathcal{R}_n$ is annihilated by p^n, and the composite $\mathcal{R}_n\xrightarrow{V^n\cdot -}\mathcal{R}_{2n}\xrightarrow{\pi}\mathcal{R}_n$ vanishes by definition.

The above proposition shows that $\mathcal{DG}r_c^-(\mathcal{R})$ is a triangulated subcategory of $\mathcal{DG}r^-(\mathcal{R})$. In particular, coherent $\mathcal{R}$-modules form a thick abelian subcategory of the category of left graded $\mathcal{R}$-modules. Consequently, $\mathcal{DG}r_c(\mathcal{R})$ (resp. $\mathcal{DG}r_c^b(\mathcal{R})$) is also a triangulated subcategory of $\mathcal{DG}r(\mathcal{R})$ (resp. $\mathcal{DG}r^b(\mathcal{R})$).

Definition 2.33. Let $M \in \mathcal{DG}r_c(\mathcal{R})$. By Proposition 2.32, after passing to the colimit we know that for every pair (i,j) the group $\mathrm{H}^j(\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M)^i$ is a finite-dimensional k-vector space. We denote its dimension by $h^{i,j}(M)$ and call it the (i,j)-th *Hodge number* of M.

The above definition is justified by the following result.

Theorem 2.34 ([IR83, Théorème II.1.2])**.** *For any smooth scheme X over the perfect field k, there are functorial identifications*

$$W_n\Omega^\bullet_{X/k} \cong \mathcal{R}_n \otimes_{\mathcal{R}} W\Omega^\bullet_{X/k} \cong \mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} W\Omega^\bullet_{X/k}.$$

We can now give the promised proof of Theorem 2.22.

Proof of Theorem 2.22. By Proposition 2.32 and Theorem 2.34, the claim follows immediately from the finiteness of the Hodge cohomology of the smooth proper variety X/k. □

Next, let us introduce some homological algebra concerning the functor $\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} - \colon \mathcal{DG}r(\mathcal{R}) \to \mathcal{DG}r(k[d]/d^2)$.

Lemma 2.35 (c.f. [Eke84, Lemma III.5.6.1])**.** *Let $M \in \mathcal{DG}r(\mathcal{R})$. Then there is a natural map*

$$\mathcal{R}/p \otimes^{\mathrm{L}}_{\mathcal{R}} M \to \mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M$$

in $\mathcal{DG}r(k[d]/d^2)$, which becomes an isomorphism after applying the functor Tot *from Notation 2.8.*

Proof. There is a natural surjection $\mathcal{R}/p \to \mathcal{R}_1$ of graded $(k[d]/d^2, \mathcal{R})$-bimodules, giving rise to the map in the statement. Let us compute its kernel: it is $\bigoplus_{n\geqslant 1} k\cdot V^n$ in grading 0 and $\bigoplus_{n\geqslant 1} k\cdot dV^n$ in grading 1. Note that the right $\mathcal{R}$-module structure has the action of d given by 0. Hence, as a graded $(k[d]/d^2, \mathcal{R})$-bimodule, we can describe it as $k[d]/d^2 \otimes_k \bigoplus_{n\geqslant 1} k \cdot V^n$. Therefore, the fiber of $\mathcal{R}/p \otimes^{\mathrm{L}}_{\mathcal{R}} M \to \mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M$ becomes $k[d]/d^2 \otimes_k \Big((\bigoplus_{n\geqslant 1} k \cdot V^n) \otimes^{\mathrm{L}}_{\mathcal{R}} M \Big)$. This object vanishes after applying Tot, thanks to Proposition 2.11. □

Construction 2.36. As a consequence of Lemma 2.35, for any grading-left bounded object $M \in \mathcal{DG}r^l(\mathcal{R})$, we obtain the spectral sequence

$$E_1^{i,j} = \mathrm{H}^j(\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M)^i \Rightarrow \mathrm{H}^{i+j}(k \otimes^{\mathrm{L}}_{W} \mathrm{Tot}(M)),$$

which we refer to as the Hodge–de Rham spectral sequence for M.

Let us also record a result of Ekedahl concerning the homological algebra of $\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} -$. Note that the original statement in loc. cit. contains a typo; here we state the corrected version based on the proof given there.

Proposition 2.37 ([Eke85, Proposition I.1.1])**.** *Let A be a thick subcategory of W-modules (ungraded) stable under σ_*, and let $M \in \mathcal{DG}r(\mathcal{R})$.*

(1) *If M is either grading-left bounded or bounded from above, and for some $r, s \in \mathbb{Z}$, $\mathrm{H}^j(\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M)^i \in A$ for all*

$$\begin{aligned}(i,j) \in\ & \big\{(i,j) \in \mathbb{Z}^2 \mid i+j = r,\ j \geqslant s\big\} \\ & \cup \big\{(i,j) \in \mathbb{Z}^2 \mid i+j = r+1,\ j \geqslant s+1\big\} \\ & \cup \big\{(i,j) \in \mathbb{Z}^2 \mid i+j = r-1,\ j \geqslant s\big\},\end{aligned}$$

then for all n, $\mathrm{H}^j(\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} M)^i \in A$ for all $(i,j) \in \mathbb{Z}^2$ such that $i+j=r$ and $j \geqslant s$.

(2) *If M is either grading-right bounded or bounded from below, and for some $r, s \in \mathbb{Z}$, $\mathrm{H}^j(\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M)^i \in A$ for all*

$$\begin{aligned}(i,j) \in \ & \big\{(i,j) \in \mathbb{Z}^2 \mid i+j=r,\ j \leqslant s\big\} \\ & \cup\big\{(i,j) \in \mathbb{Z}^2 \mid i+j=r+1,\ j \leqslant s\big\} \\ & \cup\big\{(i,j) \in \mathbb{Z}^2 \mid i+j=r-1,\ j \leqslant s-1\big\},\end{aligned}$$

then for all n, $\mathrm{H}^j(\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} M)^i \in A$ for all $(i,j) \in \mathbb{Z}^2$ such that $i+j=r$ and $j \leqslant s$.

In particular, by taking $A=\{0\}$ we obtain the following consequence: suppose M is bounded in one of the four directions. Then $M=0$ if and only if M is complete and $\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M = 0$.

Proposition 2.38. *Let $M \in \mathcal{DG}r_c(\mathcal{R})$, and assume that it is grading left-bounded. Suppose $\mathrm{H}^j(\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M)^i = 0$ for all $i+j<d$. Then $\mathrm{H}^j(M)^i = 0$ for all $i+j<d$ as well.*

Proof. By Proposition 2.37, we see that $\mathrm{H}^j(M)^i = 0$ for all $i+j<d-1$. It therefore suffices to show that $\mathrm{H}^j(M)^i = 0$ for all $i+j=d-1$.

By assumption, there exists an integer N such that $\mathrm{H}^j(M)^a = 0$ for all $a<N$. In particular, we see that $\mathrm{H}^{(d-1-i)}(M)^i = 0$ when i is sufficiently small. We prove by induction on i that this vanishing holds for all i. Suppose it has been shown for all $i<m$; we need to prove that $\mathrm{H}^{(d-1-m)}(M)^m = 0$.

Consider the spectral sequence of left graded $k[d]/d^2$-modules:

$$E_2^{\ell,j} = \mathrm{Tor}^{\mathcal{R}}_{-\ell}(\mathcal{R}_1, \mathrm{H}^j(M)) \Rightarrow \mathrm{H}^{\ell+j}(\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M).$$

By Proposition 2.25, the above groups vanish whenever $\ell \notin [-2,0]$. Moreover, if M is grading-left bounded by D, then $\mathrm{Tor}^{\mathcal{R}}_2(\mathcal{R}_1, M)$ is grading-left bounded by $D+1$. As a result, the cokernel of the map

$$\mathrm{Tor}^{\mathcal{R}}_2(\mathcal{R}_1, \mathrm{H}^{j+1}(M)) \to \mathrm{Tor}^{\mathcal{R}}_0(\mathcal{R}_1, \mathrm{H}^j(M))$$

is a subobject of $\mathrm{H}^j(\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M)$.

Now consider $\mathrm{H}^{(d-1-m)}(\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M)^m$. Our assumption implies that it is 0. The induction hypothesis implies that $\mathrm{H}^{(d-m)}(M)^{(m-1)} = 0$. From the initial observation, we also know that $\mathrm{H}^{<(d-m)}(M)^{(m-1)} = 0$. Hence $\mathrm{H}^{(d-m)}(M)$ is grading-left bounded by m. Consequently, $\mathrm{Tor}^{\mathcal{R}}_2(\mathcal{R}_1, \mathrm{H}^{(d-m)}(M))$ is grading-left bounded by $m+1$.

Combining this with the previous paragraph, we conclude that $\mathrm{Tor}^{\mathcal{R}}_0(\mathcal{R}_1, \mathrm{H}^{(d-1-m)}(M))$ has no component in grading m. Since $\mathrm{H}^{(d-1-m)}(M)$, again by the result observed at the beginning, is grading-left bounded by m, we conclude that

$$\mathrm{Tor}^{\mathcal{R}}_0(\mathcal{R}_1, \mathrm{H}^{(d-1-m)}(M))^m = \mathrm{H}^{(d-1-m)}(M)^m / V \cdot \mathrm{H}^{(d-1-m)}(M)^m = 0.$$

Therefore, $\mathrm{H}^{(d-1-m)}(M)^m / V^n \cdot \mathrm{H}^{(d-1-m)}(M)^m = 0$ for all positive integers n. By assumption, the cohomology $\mathrm{H}^{(d-1-m)}(M)$ is coherent, which by definition implies that it is classically complete. Hence the above vanishing implies that $\mathrm{H}^{(d-1-m)}(M)^m = 0$. ☐

2.4 Hodge–Witt numbers

Below we recall several relations between Hodge–Witt numbers and Hodge numbers.

Theorem 2.39 (Crew's formula, [Cre85, Theorem 4], [Eke86, Theorem IV.3.2(2)]). *Let $M \in \mathcal{DG}r_c^b(\mathcal{R})$. For any natural number i, one has*

$$\sum_j (-1)^j h_W^{i,j} = \sum_j (-1)^j h^{i,j}. \tag{2.40}$$

Corollary 2.41 (see also [MR15, Lemma 2.5]). *For any short exact sequence of coherent $\mathcal{R}$-modules*

$$0 \to M_1 \to M \to M_2 \to 0,$$

for every $i \in \mathbb{Z}$ one has

$$T^i(M) = T^i(M_1) + T^i(M_2).$$

Consequently, let M be a coherent $\mathcal{R}$-module and let $M_\bullet$ be any filtration satisfying the conclusion of [IR83, Proposition I.2.19]. Then the multiplicity of the i-th shift of type II *in the associated graded object is independent of the choice of filtration.*

Proof. By definition, all differentials d of a coherent $\mathcal{R}$-module become zero after inverting p. Consequently, the multiplicity of a slope λ, namely $m_\lambda(\text{cœur}(M^i) \otimes K)$, is additive in short exact sequences of coherent $\mathcal{R}$-modules. It follows that the slope numbers $m^{i,j}$ and the quantities

$$\sum_{\lambda \in [0,1)} (1-\lambda) m_\lambda(\text{cœur}(M^i) \otimes K), \qquad \sum_{\lambda \in [0,1)} \lambda m_\lambda(\text{cœur}(M^{i-1}) \otimes K)$$

are additive for any short exact sequence of coherent $\mathcal{R}$-modules.

Viewing M, M_1, M_2 as objects of $\mathcal{DG}r_c^b(\mathcal{R})$, the short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ induces an exact triangle $\mathcal{R}_1 \otimes_{\mathcal{R}}^{\mathrm{L}} M_1 \longrightarrow \mathcal{R}_1 \otimes_{\mathcal{R}}^{\mathrm{L}} M \longrightarrow \mathcal{R}_1 \otimes_{\mathcal{R}}^{\mathrm{L}} M_2$ in $\mathcal{DG}r(k[d]/d^2)$. Taking the component of module degree i yields long exact sequences in cohomology. Hence the Euler characteristic $\sum_j (-1)^j h^{i,j}(M)$ is additive. By Theorem 2.39, the same holds for $\sum_j (-1)^j h_W^{i,j}(M) = \sum_j (-1)^j h^{i,j}(M)$.

For a coherent $\mathcal{R}$-module M, viewed as an object of $\mathcal{DG}r_c^b(\mathcal{R})$, the Hodge–Witt numbers are given by

$$\begin{aligned} h_W^{i,0}(M) &= \sum_{\lambda \in [0,1)} (1-\lambda) m_\lambda(\text{cœur}(M^i) \otimes K) + T^i(M), \\ h_W^{i,-1}(M) &= \sum_{\lambda \in [0,1)} \lambda m_\lambda(\text{cœur}(M^{i-1}) \otimes K) - 2T^{i-1}(M), \\ h_W^{i,-2}(M) &= T^{i-2}(M). \end{aligned}$$

Since the slope multiplicities are additive, it follows that the quantity

$$T^i(M) + 2T^{i-1}(M) + T^{i-2}(M)$$

is additive for every $i \in \mathbb{Z}$. An induction on i then shows that each T^i is additive for short exact sequences of coherent $\mathcal{R}$-modules. □

Definition 2.42. Let X be a smooth proper variety over a perfect field k of characteristic p. We define the *domino numbers* $T^{i,j}(X)$, *slope numbers* $m^{i,j}(X)$, *Hodge–Witt numbers* $h_W^{i,j}(X)$, and *Hodge numbers* $h^{i,j}(X)$ of X to be the corresponding invariants (see Definition 2.19 and Definition 2.33) of the object $R\Gamma(X, W\Omega^\bullet_{X/k}) \in \mathcal{DG}r_c^b(\mathcal{R})$.

Definition 2.43 ([IR83, Definition IV.4.6])**.** A smooth proper variety X over k is called a Hodge–Witt variety if all of its domino numbers are 0.

Theorem 2.44 ([IR83, Theorem IV.4.5])**.** *Let X be a smooth proper variety over k. Assume that X is Hodge–Witt. Then for each $n \in \mathbb{Z}$ there is a canonical decomposition $\mathrm{H}^n_{\mathrm{crys}}(X/W) \cong \bigoplus_{i+j=n} \mathrm{H}^j(X, W\Omega^i_{X/k})$, such that the Frobenius action on the left-hand side corresponds to the action of $\bigoplus p^i F$ on the right-hand side.*

Ekedahl also studied whether the Hodge–Witt numbers satisfy symmetry analogous to Hodge numbers in characteristic 0, he obtained the following:

Proposition 2.45 ([Eke86, Proposition IV.3.2, 3.3])**.** *Let X/k be a proper smooth variety of pure dimension d.*

(1) *Let $n \in \mathbb{N}$. Then $h^{i,j}_W = h^{j,i}_W$ for all $(i,j) \in \mathbb{N}^2$ such that $i + j = n$ if and only if $T^{i,j} = T^{j-2,i+2}$ for all $(i,j) \in \mathbb{N}^2$ such that $i + j = n$.*

(2) *The equivalent conditions in* (1) *is satisfied whenever $n \leqslant 2$ or $d \leqslant 3$.*

3 Deeper structures on $\mathcal{R}$-complexes

In this section, we review finer structures on $\mathcal{DG}r(\mathcal{R})$ and $\mathcal{DG}r_c(\mathcal{R})$, as exhibited by Ekedahl [Eke84, Eke85, Eke86].

3.1 Duality

The goal of this subsection is to prove Corollary 3.21. Surprisingly, we need to use a dualizing functor introduced by Ekedahl in [Eke84, Chapter III], which we first review below.[6]

Definition 3.1 ([Eke84, Page. 189])**.** For each n, let $\rho : \mathcal{R}_n \to \mathcal{R}_{n+1}$ be the unique injective map such that the following diagram commutes (the map ρ is unique because π and p have the same kernel):

$$\begin{array}{ccc} \mathcal{R}_{n+1} & \xrightarrow{\pi} & \mathcal{R}_n \\ & \searrow^{p} & \downarrow{\scriptstyle \rho} \\ & & \mathcal{R}_{n+1} \end{array}$$

Definition 3.2 ([Eke84, Page. 205])**.** For each $n \in \mathbb{N}$, let

$$\breve{\mathcal{R}}_n \coloneqq \mathrm{Hom}_{W_n}(\mathcal{R}_n, W_n),$$

viewed as a graded left $\mathcal{R}$-module via the graded right $\mathcal{R}$-module structure of the source. Let $\rho^* : \breve{\mathcal{R}}_{n+1} \to \breve{\mathcal{R}}_n$ be the map sending a functional f to $f \circ \rho$ (identifying the p^n-torsion in W_{n+1} with "$p \cdot W_n$"). We view $\{\breve{\mathcal{R}}_n, \rho^*\}$ as a projective system of graded left $\mathcal{R}$-modules. We equip it with another graded left $\mathcal{R}$-module structure as follows:

[6] Note that in [Eke85, Definition I.6.1] Ekedahl also defined another dualizing functor, and showed that these two dualizing functors agree on $\mathcal{DG}r^b_c(\mathcal{R})$ (see [Eke85, Proposition III.1.6]).

- Define the action of $F\colon \mathrm{Hom}_{W_{n+1}}(\mathcal{R}_{n+1}, W_{n+1}) \to \mathrm{Hom}_{W_n}(\mathcal{R}_n, W_n)$ by the rule that for any functional $f \in \mathrm{Hom}_{W_{n+1}}(\mathcal{R}_{n+1}, W_{n+1})$ and any $r \in \mathcal{R}_n$, we have the following equality in W_{n+1}:

$$p \cdot F(f)(r) = \sigma(f(V \cdot r)),$$

where $W_n \xhookrightarrow{p\cdot -} W_{n+1}$.

- Similarly, define the action of $V\colon \mathrm{Hom}_{W_n}(\mathcal{R}_n, W_n) \to \mathrm{Hom}_{W_{n+1}}(\mathcal{R}_{n+1}, W_{n+1})$ by the rule that for any functional $f \in \mathrm{Hom}_{W_n}(\mathcal{R}_n, W_n)$ and any $r \in \mathcal{R}_{n+1}$, we have the following equality in W_{n+1}:

$$\sigma(V(f)(r)) = p \cdot f(F \cdot r).$$

- Lastly, define the action of $d\colon \breve{\mathcal{R}}_n \to \breve{\mathcal{R}}_n(1) \coloneqq \mathrm{Hom}_{W_n}(\mathcal{R}_n(-1), W_n)$ by the rule that for any functional $f \in \mathrm{Hom}_{W_n}(\mathcal{R}_n, W_n)$ and any $r \in \mathcal{R}_n$, we have the equality in W_n:

$$d(f)(r) = f(d \cdot r).$$

Since left multiplication commutes with right multiplication, one easily checks that the above actions respect the individual graded left $\mathcal{R}$-module structures. Finally, we define

$$\breve{\mathcal{R}} \coloneqq \lim_{n \in \mathbb{N}} \breve{\mathcal{R}}_n,$$

viewed as a graded left $\mathcal{R} \otimes_{\mathbb{Z}_p} \mathcal{R}$-module. We follow Ekedahl's notation: the action of the first copy of $\mathcal{R}$ is induced by the right $\mathcal{R}$-module structure on $\mathcal{R}_n$, and the second copy acts by taking the inverse limit of the actions defined above.

Notation 3.3. In order to minimize confusion, let us denote the inclusion $\mathcal{R} \to \mathcal{R} \otimes_{\mathbb{Z}_p} \mathcal{R}$ via the first (resp. second) factor by i (resp. j). When we wish to emphasize that we view $\breve{\mathcal{R}}$ as a graded left $\mathcal{R}$-module via the first (resp. second) action of $\mathcal{R}$, we denote it by $i_*(\breve{\mathcal{R}})$ (resp. $j_*(\breve{\mathcal{R}})$).

Definition 3.4 (Ekedahl's dualizing functor [Eke84, Definition III.2.8, Proposition III.3.2])**.** Define a contravariant functor from $\mathcal{DG}r(\mathcal{R})$ to itself as follows. Given $M \in \mathcal{DG}r(\mathcal{R})$, let

$$D_{\mathcal{R}}(M) \coloneqq \mathrm{RHom}_{\mathcal{R}}(M, i_*(\breve{\mathcal{R}})).$$

Here the left $\mathcal{R}$-action comes from the second $\mathcal{R}$-action on $\breve{\mathcal{R}}$.

In order to examine the properties of the dualizing functor, we need to explicate $\breve{\mathcal{R}}_n$ and $\breve{\mathcal{R}}$. To that end, let us consider the following W-linear graded functional $\psi : \mathcal{R}(-1) \to W$ defined by

$$\begin{cases} \psi(F^n) = \psi(V^n) = 0 \text{ for all } n \geqslant 0, \\ \psi(dV^n) = \psi(F^n d) = 0 \text{ for all } n \geqslant 1, \\ \psi(d) = 1. \end{cases}$$

Next we define a graded left $\mathcal{R}$-module as follows:

$${}_n\overline{\mathcal{R}} \coloneqq \left\{ \sum_{m \geqslant 0} V^m a_m + \sum_{1 \leqslant m \leqslant n-1} b_m F^m + \sum_{m \geqslant 0} dV^m c_m + \sum_{1 \leqslant m \leqslant n-1} e_m F^m d \;\middle|\; a_m, c_m \in W/p^n,\ b_m, e_m \in W/p^{n-m} \right\}.$$

We now define a pairing $\mathcal{R}_n \times_n \overline{\mathcal{R}} \to W_n$ by sending $(r, s) \mapsto$ "$\psi(r \cdot s)$". Namely, we lift r to an element $\widetilde{r}$ in $\mathcal{R}$; the result is the coefficient "c_0" of the element $\widetilde{r} \cdot s \in_n \overline{\mathcal{R}}$. A direct verification shows the following.

Lemma 3.5. *The above construction does not depend on the choice of the lift $\widetilde{r}$ and induces an isomorphism of graded left $\mathcal{R}$-modules*

$$\alpha_n \colon {}_n\overline{\mathcal{R}} \xrightarrow{\cong} \breve{\mathcal{R}}_n.$$

Moreover, the map $\alpha_n^{-1} \circ \rho^* \circ \alpha_{n+1} \colon {}_{n+1}\overline{\mathcal{R}} \to_n \overline{\mathcal{R}}$ is the natural projection obtained by reducing each coefficient modulo a smaller power of p. In particular, the transition maps ρ^* are all surjective.

Proposition 3.6 ([Eke84, Proposition III.3.5])**.** *The isomorphisms $\{\alpha_n\}$ in Lemma 3.5 give rise to an identification*[7]

$$\alpha\colon \big(\prod_{m\geqslant 0} V^m \cdot W\big) \oplus \big(\prod_{m\geqslant 1} W \cdot F^m\big) \oplus \big(\prod_{m\geqslant 0} dV^m \cdot W\big) \oplus \big(\prod_{m\geqslant 1} W \cdot F^m d\big) \xrightarrow{\cong} \breve{\mathcal{R}}.$$

Under the above identification, the first left $\mathcal{R}$-action on $\breve{\mathcal{R}}$ corresponds to left multiplication, while the second $\mathcal{R}$-action on $\breve{\mathcal{R}}$ corresponds to right multiplication. This defines a left $\mathcal{R}^{\mathrm{op}}$-action, which can then be viewed as a left $\mathcal{R}$-action via the isomorphism $\iota\colon \mathcal{R} \xrightarrow{\cong} \mathcal{R}^{\mathrm{op}}$ sending F, d, V and $a \in W$ to V, d, F and $a \in W$, respectively.

Corollary 3.7 ([Eke84, Corollary III.3.5.1])**.** *The projection $\breve{\mathcal{R}} \to \breve{\mathcal{R}}_n$ induces an isomorphism:*

$$\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} j_* \breve{\mathcal{R}} \xrightarrow{\cong} \breve{\mathcal{R}}_n$$

of graded left $\mathcal{R}$-modules. Here the action on the left hand side is given by $r \cdot (a \otimes b) \coloneqq a \otimes (i(r) \cdot b)$.

Now we can deduce the first property of Ekedahl's dualizing functor:

Proposition 3.8. *There is a natural equivalence of contravariant functors*

$$\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} D_{\mathcal{R}}(-) \cong D_{W_n}(\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} -)\colon \mathcal{DG}r(\mathcal{R})^{\mathrm{op}} \to \mathcal{DG}r(W_n[d]/d^2),$$

where $D_{W_n} \coloneqq \mathrm{RHom}_{W_n}(-, W_n)$ is the W_n-linear dualizing functor.

Proof. Let us compute:

$$\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} D_{\mathcal{R}}(-) \cong \mathrm{RHom}_{\mathcal{R}}(-, \mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} j_*\breve{\mathcal{R}}) \cong \mathrm{RHom}_{\mathcal{R}}(-, \mathrm{RHom}_{W_n}(\mathcal{R}_n, W_n)) \cong \mathrm{RHom}_{W_n}(\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} -, W_n).$$

Here the first identification follows from the fact that $\mathcal{R}_n$ is a perfect right $\mathcal{R}$-complex, the second identification uses Corollary 3.7, and the last identification is the tensor–Hom adjunction. ☐

We also know that Ekedahl's dualizing functor automatically takes values in $\widehat{\mathcal{DG}r(\mathcal{R})}$ (see Construction 2.24).

Proposition 3.9. *The functor $D_{\mathcal{R}}$ defines a contravariant functor from $\mathcal{DG}r(\mathcal{R})$ to $\widehat{\mathcal{DG}r(\mathcal{R})}$.*

Proof. By Proposition 2.28, it suffices to show that the natural map $D_{\mathcal{R}}(M) \to \widehat{D_{\mathcal{R}}(M)}$ is an equivalence. Since $\mathcal{R}_n$ is a perfect right $\mathcal{R}$-complex, we have

$$R\lim_n \big(\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} D_{\mathcal{R}}(-)\big) \cong R\lim_n \mathrm{RHom}_{\mathcal{R}}(-, \mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} j_*\breve{\mathcal{R}}) \cong \mathrm{RHom}_{\mathcal{R}}\Big(-, R\lim_n \mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} j_*\breve{\mathcal{R}}\Big).$$

[7] Following Ekedahl, we write the coefficients from W in this somewhat awkward way, so that the bijection β from Construction 3.16 has a convenient form.

The natural arrow is induced by the natural map $i_*(\breve{\mathcal{R}}) \to R\lim_n \mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} j_*\breve{\mathcal{R}}$, which is an isomorphism of graded left $\mathcal{R}$-modules due to Corollary 3.7, the surjectivity of ρ^* (Lemma 3.5), and the definition of $\breve{\mathcal{R}}$. □

A common feature of all known dualizing functors is the existence of a natural transformation from the identity functor to the double dualizing functor. Our task is to exhibit such a natural transformation in the context of Ekedahl's dualizing functor, following [Eke84, Section IV.1]. To that end, we begin with a brief digression.

Notation 3.10. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. We denote $\mathcal{C}^- := \bigcup_{n\in\mathbb{Z}} \mathcal{C}^{\leqslant n}$, and call it the full subcategory of right-bounded objects.

Notation 3.11. Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories equipped with t-structures. We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *left t-bounded* if there exists an integer N such that $F(\mathcal{C}^{\geq 0}) \subset \mathcal{D}^{\geq N}$. The notion of a *right t-bounded* functor is defined analogously. A functor is called *t-bounded* if it is both left and right t-bounded. We denote $\mathrm{Fun}^{\mathrm{lb}}_{\mathrm{ex}}(\mathcal{C},\mathcal{D})$ the full sub-category of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ spanned by left t-bounded exact functors.

Example 3.12. Let $\mathcal{C} = \mathcal{D} = \mathcal{DG}r(\mathcal{R})$, equipped with the standard t-structure, which is both left and right complete. The identity functor is clearly t-exact, hence t-bounded. We claim that the double dual functor $D_{\mathcal{R}} \circ D_{\mathcal{R}}$ is also t-bounded: Using Proposition 3.8 and Proposition 3.9, we obtain

$$D_{\mathcal{R}}(D_{\mathcal{R}}(M)) = \mathrm{Rlim}_n\, D_{W_n}\big(D_{W_n}(\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} M)\big).$$

Thus the claim follows from the fact that the following functors: $(\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} -)$, double dual over W_n, and $\mathrm{Rlim}_{n\in\mathbb{N}}$ are all t-bounded.

Proposition 3.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories equipped with t-structures. Assume that $\mathcal{D}$ is right complete. Then restriction along $\mathcal{C}^- \subset \mathcal{C}$ induces an equivalence: $\mathrm{Fun}^{\mathrm{lb}}_{\mathrm{ex}}(\mathcal{C},\mathcal{D}) \xrightarrow{\cong} \mathrm{Fun}^{\mathrm{lb}}_{\mathrm{ex}}(\mathcal{C}^-,\mathcal{D})$.*

Proof. First we claim that every left t-bounded exact functor $F\colon \mathcal{C}^- \to \mathcal{D}$ admits a left Kan extension. According to [Lur09, Lemma 4.3.2.13], it suffices to check that for any object $C \in \mathcal{C}$ the colimit $\mathrm{colim}_{X\in\mathcal{C}^-_{/C}} F(X)$ exists in $\mathcal{D}$. Since the sequence of truncations $\{\tau^{\leqslant n}C\}_{n\in\mathbb{Z}}$ is cofinal in $\mathcal{C}^-_{/C}$, it suffices to show that the colimit $\mathrm{colim}_{n\in\mathbb{Z}} F(\tau^{\leqslant n}C)$ exists in $\mathcal{D}$. This follows from the assumption that F is left t-bounded and exact, together with the right completeness of $\mathcal{D}$.

One checks directly that the left Kan extension $\mathrm{Lan}(F)(C) := \mathrm{colim}_{n\in\mathbb{Z}} F(\tau^{\leqslant n}C)$ is again left t-bounded and exact. Therefore, by [Lur09, Proposition 4.3.2.15], we obtain a functor

$$\mathrm{Fun}^{\mathrm{lb}}_{\mathrm{ex}}(\mathcal{C}^-,\mathcal{D}) \xrightarrow{\mathrm{Lan}(-)} \mathrm{Fun}^{\mathrm{lb}}_{\mathrm{ex}}(\mathcal{C},\mathcal{D})$$

sending F to its left Kan extension.

Similar in the proof of [Lur09, Proposition 4.3.2.17], using [Lur09, Lemma 4.3.2.12], one sees that $\mathrm{Lan}(-)$ is left adjoint to the restriction functor. Moreover, since the restriction of left Kan extension is identity, the functor $\mathrm{Lan}(-)$ is fully faithful.

Finally, it suffices to show that $\mathrm{Lan}(-)$ is essentially surjective. This amounts to proving that for every left t-bounded exact functor $G\colon \mathcal{C} \to \mathcal{D}$, the natural map $\mathrm{colim}_{n\in\mathbb{Z}} G(\tau^{\leqslant n}C) \longrightarrow G(C)$ is an equivalence. This again follows from the assumption that G is left t-bounded and exact and that $\mathcal{D}$ is right complete. □

Proposition 3.14. *Let $\mathcal{D}$ be a stable ∞-category equipped with a t-structure that is both left and right complete. Let K be a directed graph, and let $\{F_k\}$ be a collection of left t-bounded, uniformly right t-bounded, exact functors from $\mathcal{DG}r(\mathcal{R})$ to $\mathcal{D}$ indexed by the vertices of K. For any full subcategory $\mathcal{C} \subset$*

$\mathcal{DG}r(\mathcal{R})$, we denote by $\mathrm{Fun}_F(K\times\mathcal{C},\mathcal{D})$ the fiber of $\{F_k|_{\mathcal{C}}\}$ under the restriction to vertices map. Let $\mathcal{P}roj\mathcal{G}r(\mathcal{R})\subset\mathcal{DG}r(\mathcal{R})$ denote the full subcategory of projective graded left $\mathcal{R}$-modules. Then the restriction map induces an equivalence

$$\mathrm{Fun}_F(K\times\mathcal{DG}r(\mathcal{R}),\mathcal{D})\xrightarrow{\cong}\mathrm{Fun}_F(K\times\mathcal{P}roj\mathcal{G}r(\mathcal{R}),\mathcal{D}).$$

Proof. By Proposition 3.13 and the assumption that the functors $\{F_k\}$ are all left t-bounded and exact, the restriction map induces an equivalence $\mathrm{Fun}_F(K\times\mathcal{DG}r(\mathcal{R}),\mathcal{D})\xrightarrow{\cong}\mathrm{Fun}_F(K\times\mathcal{DG}r^-(\mathcal{R}),\mathcal{D})$.

Since the collection of functors $\{F_k\}$ is uniformly right t-bounded, after a finite shift we may assume without loss of generality that all of them are right t-exact (this does not affect the condition that they are left t-bounded). As $\mathcal{DG}r^-(\mathcal{R})$ is left complete and right bounded, and $\mathcal{D}$ is assumed to be left complete, the statement now follows from the combination of [Lur17, Theorem 1.3.3.8 and Lemma 1.3.3.11]. □

Lemma 3.15. *For any projective graded left $\mathcal{R}$-module M, the dual $D_{\mathcal{R}}(M)$ lies in $\mathcal{DG}r^{[0,0]}(\mathcal{R})$, and the double dual $D_{\mathcal{R}}(D_{\mathcal{R}}(M))$ lies in $\mathcal{DG}r^{[0,1]}(\mathcal{R})$.*[8]

Proof. Such an M is a direct summand of a graded free[9] left $\mathcal{R}$-module, so we may assume that M is graded free. Writing M as a filtered colimit of finite direct sums, the dual $D_{\mathcal{R}}(M)$ becomes a cofiltered limit of finite products of $\breve{\mathcal{R}}(-n_i)$, and hence lies in cohomological degree 0. Using Proposition 3.8 and Proposition 3.9, we obtain

$$D_{\mathcal{R}}(D_{\mathcal{R}}(M))=\mathrm{Rlim}_n\, D_{W_n}\big(D_{W_n}(\mathcal{R}_n\otimes^{\mathrm{L}}_{\mathcal{R}}M)\big).$$

The claim now follows from the facts that $\mathcal{R}_n\otimes^{\mathrm{L}}_{\mathcal{R}}M$ lies in cohomological degree 0, the W_n-linear double dual is t-exact, and the functor $\mathrm{Rlim}_{n\in\mathbb{N}}$ has cohomological amplitude contained in $[0,1]$. □

Construction 3.16 (Natural map to the double dual)**.** We now construct a natural transformation

$$\mathrm{id}\xrightarrow{\mathrm{ev}_{\mathcal{R}}}D_{\mathcal{R}}\circ D_{\mathcal{R}}.$$

In view of Example 3.12 and Proposition 3.14, it suffices to construct a natural transformation $\mathrm{id}\to D_{\mathcal{R}}\circ D_{\mathcal{R}}$ between functors from $\mathcal{P}roj\mathcal{G}r(\mathcal{R})$ to $\mathcal{DG}r(\mathcal{R})$. By Lemma 3.15, it suffices to construct natural maps of graded left $\mathcal{R}$-modules

$$M\to\text{“}j_*\text{”}\mathrm{Hom}_{\mathcal{R}}(\text{“}j_*\text{”}\mathrm{Hom}_{\mathcal{R}}(M,i_*(\breve{\mathcal{R}})),i_*(\breve{\mathcal{R}}))$$

for projective graded left $\mathcal{R}$-modules. Indeed, maps from M to an object in $\mathcal{DG}r^{[0,n]}(\mathcal{R})$ automatically factor through its H^0. Here we write “j_*” to remind the reader that the $\mathcal{R}$-module structure arises from the second action on the target $\breve{\mathcal{R}}$. Unwinding the definition, this is equivalent to constructing natural maps of graded left $\mathcal{R}$-modules

$$\text{“}j_*\text{”}\mathrm{Hom}_{\mathcal{R}}(M,i_*(\breve{\mathcal{R}}))\longrightarrow\text{“}i_*\text{”}\mathrm{Hom}_{\mathcal{R}}(M,j_*(\breve{\mathcal{R}})).$$

Finally, this is achieved by exhibiting a bijection $\breve{\mathcal{R}}\xrightarrow[\cong]{\beta}\breve{\mathcal{R}}$ that swaps the $\mathcal{R}$-actions via i and j. Using the

[8] In fact, with a more involved argument, one can show that $D_{\mathcal{R}}(D_{\mathcal{R}}(M))$ actually lives in cohomological degree 0. However, since the present cohomological estimate is sufficient for our purposes, we do not pursue this stronger statement.

[9] By this we mean a module of the form $M=\bigoplus_{i\in I}\mathcal{R}(n_i)$ for some set I.

isomorphism α from Proposition 3.6, this map is given by

$$\sum_{m\geqslant 0} V^m a_m + \sum_{m\geqslant 1} b_m F^m + \sum_{m\geqslant 0} dV^m c_m + \sum_{m\geqslant 1} e_m F^m d \mapsto \sum_{m\geqslant 1} V^m b_m + \sum_{m\geqslant 0} a_m F^m + \sum_{m\geqslant 1} dV^m e_m + \sum_{m\geqslant 0} c_m F^m d.$$[10]

Concretely, the map $M \to$ "j_*"$\mathrm{Hom}_{\mathcal{R}}($"$j_*$"$\mathrm{Hom}_{\mathcal{R}}(M, i_*(\breve{\mathcal{R}})), i_*(\breve{\mathcal{R}}))$ sends an element $m \in M$ to the functional that associates to $f \in \mathrm{Hom}_{\mathcal{R}}(M, i_*(\breve{\mathcal{R}}))$ the element $\beta(f(m)) \in \breve{\mathcal{R}}$.

After defining the natural transformation, our next task is to show its compatibility with the W_n-linear dualizing functor via $\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} -$.

Proposition 3.17 ([Eke84, Lemma IV.1.5])**.** *There is a commutative diagram of functors from $\mathcal{DG}r(\mathcal{R})$ to $\mathcal{DG}r(W_n[d]/d^2)$:*

$$\begin{array}{ccc}
\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} \mathrm{id} & \xrightarrow{\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} (\mathrm{ev}_{\mathcal{R}})} & \mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} (D_{\mathcal{R}}(D_{\mathcal{R}}(-)) \\
\Vert & & \downarrow \cong \\
 & & D_{W_n}(\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} D_{\mathcal{R}}(-)) \\
\Vert & & \downarrow \cong \\
\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} (-) & \xrightarrow{\mathrm{ev}_{W_n}} & D_{W_n}(D_{W_n}(\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} (-))).
\end{array}$$

Here $\mathrm{ev}_{W_n} : \mathrm{id} \to D_{W_n} \circ D_{W_n}$ denotes the usual natural transformation between endofunctors of $\mathcal{DG}r(W_n[d]/d^2)$, and the identifications in the right column come from Proposition 3.8.

Proof. By Proposition 2.25 and Example 3.12, all functors involved are t-bounded. Using Proposition 3.14, it therefore suffices to exhibit such a commutative diagram with all functors viewed as defined only on projective graded left $\mathcal{R}$-modules. The values of all functors on projective graded left $\mathcal{R}$-modules are concentrated in cohomological degree 0: For the right column, it suffices to note that $D_{W_n} \circ D_{W_n}$ is t-exact. Thus we are merely verifying a compatibility condition rather than specifying additional data. Finally, the diagram commutes as explained by Ekedahl in *loc. cit.*. □

Remark 3.18. For the convenience of the reader, we briefly explain the commutativity above. First, we may reduce to the case where M is graded free. By writing M as a colimit of graded finite free modules, we may further reduce to the case where M is a graded shift of $\mathcal{R}$. Finally, since every arrow respects the graded structure, we may reduce to the case $M = \mathcal{R}$. In this case, after unwinding the definitions, the commutativity claim follows from the formula $\psi(r \cdot s) = \psi(\beta(s) \cdot \iota(r))$, where $r \in \mathcal{R}$, s is a power series as in Proposition 3.6, ι is the isomorphism $\mathcal{R} \cong \mathcal{R}^{\mathrm{op}}$ described in Proposition 3.6, β is the bijection introduced in Construction 3.16, and ψ is the linear functional given by taking the coefficient c_0 in the series expansion in Proposition 3.6.

From the above discussion, Ekedahl draws the following conclusion.

Proposition 3.19 ([Eke84, Proposition IV.1.1])**.** *The dualizing functor D maps $\mathcal{DG}r^b_c(\mathcal{R})$ to itself, and the evaluation transformation $\mathrm{ev} : \mathrm{id} \to D(D(-))$ is an isomorphism on $\mathcal{DG}r^b_c(\mathcal{R})$.*

We now generalize Ekedahl's result. Let n be a positive integer. Denote by $\mathcal{DG}r^{\mathcal{R}_n\text{-finite}}(\mathcal{R})$ the full subcategory of $\mathcal{DG}r(\mathcal{R})$ spanned by objects M such that $\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} M \in \mathcal{DG}r^-(W_n)$ obtained by forgetting grading and the action of d has finitely generated cohomology.

[10]See [Eke84, Proposition III.3.5].

Corollary 3.20. *Ekedahl's dualizing functor $D_{\mathcal{R}}$ preserves $\mathcal{DG}r^{\mathcal{R}_n\text{-finite}}(\mathcal{R})$. Moreover, if $M \in \mathcal{DG}r^{\mathcal{R}_n\text{-finite}}(\mathcal{R})$, then $\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} \mathrm{ev}_{\mathcal{R}} : \mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} M \longrightarrow \mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} D_{\mathcal{R}}(D_{\mathcal{R}}(M))$ is an equivalence. In particular, Ekedahl's dualizing functor $D_{\mathcal{R}}$ restricts to an anti-equivalence on $\widehat{\mathcal{DG}r}(\mathcal{R}) \cap \bigcap_{n\geq 1} \mathcal{DG}r^{\mathcal{R}_n\text{-finite}}(\mathcal{R})$.*

Proof. The first claim follows immediately from Proposition 3.8. The second claim follows immediately from Proposition 3.17. For the last claim, it suffices to check that for $M \in \widehat{\mathcal{DG}r}(\mathcal{R}) \cap \bigcap_{n\geq 1} \mathcal{DG}r^{\mathcal{R}_n\text{-finite}}(\mathcal{R})$, the map $M \xrightarrow{\mathrm{ev}_{\mathcal{R}}} D_{\mathcal{R}}(D_{\mathcal{R}}(M))$ is an equivalence. By the assumption that M is complete and by Proposition 3.9, we are reduced to the second claim. ☐

Corollary 3.21. *Let $M \in \mathcal{DG}r(\mathcal{R})$ be a complete object. Assume that M is bounded either from above or from below. Then the following conditions are equivalent:*

(1) *$M \in \mathcal{DG}r_c(\mathcal{R})$;*

(2) *for all n, the object $\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} M \in \mathcal{DG}r(W_n)$, obtained by forgetting the grading and the action of d, has finitely generated cohomology;*

(3) *the object $\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M \in \mathcal{DG}r(k)$, obtained by forgetting the grading and the action of d, has finite-dimensional cohomology.*

Moreover, the functor $D_{\mathcal{R}}$ induces an anti-equivalence between $\mathcal{DG}r_c^+(\mathcal{R})$ and $\mathcal{DG}r_c^-(\mathcal{R})$.

Proof. When M is bounded from above, this is Proposition 2.32. From now on, we assume that M is bounded from below. It is clear that (1) implies (2), and that (2) implies (3). Below we show that (3) implies (1).

First, similarly to the reasoning in Example 3.12, we note that $D_{\mathcal{R}}$ is also t-bounded. Indeed, examining the argument there shows that $D_{\mathcal{R}}(M) \in \mathcal{DG}r^{[0,3]}(\mathcal{R})$ for any $M \in \mathcal{DG}r^{[0,0]}(\mathcal{R})$.

Using the first statement in Corollary 3.20 and the case where M is bounded above, we deduce that $D_{\mathcal{R}}(M) \in \mathcal{DG}r_c^-(\mathcal{R})$. Then Proposition 3.19, together with the t-boundedness of $D_{\mathcal{R}}$, implies that $D_{\mathcal{R}}(D_{\mathcal{R}}(M)) \in \mathcal{DG}r_c^+(\mathcal{R})$. The desired conclusion now follows from Corollary 3.20. ☐

3.2 Ekedahl's star product

The goal of this subsection is to review Ekedahl's star product. In [Eke85], Ekedahl defined the star product $*$ which "corresponds to the correct Künneth formula" for the de Rham–Witt cohomology $R\Gamma(X, W\Omega^{\bullet}_{X/k})$. Below we review his theory.

Construction 3.22 ([Eke85, Definition 1.3.1])**.** Let M and N be two left graded $\mathcal{R}$-modules, the star product $M{*}N$ is defined to be the left graded $\mathcal{R}$-module generated by homogeneous elements of the form $m{*}n$ where m and n range over all homogeneous elements of M and N respectively (with $m{*}n$ of grading $\deg(m) + \deg(n)$), subject to the following relations (where λ ranges over $W(k)$):

$$
\begin{aligned}
(Vm){*}n = V(m{*}Fn), &\quad m{*}(Vn) = V(Fm{*}n),\\
F(m{*}n) =& (Fm){*}(Fn),\\
d(m{*}n) =& (dm){*}n + (-1)^{\deg(m)} m{*}(dn),\\
(m_1 + m_2){*}n = m_1{*}n + m_2{*}n, &\quad (\lambda m){*}n = \lambda(m{*}n),\\
m{*}(n_1 + n_2) = m{*}n_1 + m{*}n_2, &\quad m{*}(\lambda n) = \lambda(m{*}n).
\end{aligned}
$$

This construction comes equipped with canonical isomorphisms $M{*}N \xrightarrow{\cong} N{*}M$ sending homogeneous elements $m{*}n$ to $(-1)^{\deg(m)\cdot\deg(n)}n{*}m$ ([Eke85, Page. 71, Equation (3.4)]). Moreover, let $W = W(k)$ be the graded left $\mathcal{R}$-module placed in grading 0, with F and V acting as usual Witt vector Frobenius and Verschiebung, and $d = 0$. Then one checks that $M \xrightarrow{m\mapsto m*1} M{*}W$ is an isomorphism. In this way we obtain a symmetric monoidal structure on the category of graded left $\mathcal{R}$-modules.

By definition, it is clear that $-{*}M$ is a right-exact functor for any graded left $\mathcal{R}$-module M. Here are some properties of this symmetric monoidal structure.

Example 3.23 ([Eke85, Proposition 3.2])**.** The graded left $\mathcal{R}$-module $\mathcal{R}{*}\mathcal{R}$ is a graded free module generated by the following two sets of homogeneous elements:

$$\begin{aligned} &\text{grading } 0: F^i{*}1,\ i \geqslant 0; \quad 1{*}F^i,\ i \geqslant 1; \\ &\text{grading } 1: F^i d{*}1,\ i \geqslant 0; \quad 1{*}F^i d,\ i \geqslant 1. \end{aligned}$$

Corollary 3.24 ([Eke85, Corollary 3.2.3])**.** *For any left graded $\mathcal{R}$-module M, the underlying graded W-module (obtained by forgetting the actions of F, V and d) $\mathcal{R}{*}M$ decomposes as:*

$$\mathcal{R}{*}M \cong \Big(\bigoplus_{i>0} V^i(1{*}M)\Big) \oplus \Big(\bigoplus_{i\geqslant 0} F^i{*}M\Big) \oplus \Big(\bigoplus_{i>0} dV^i(1{*}M)\Big) \oplus \Big(\bigoplus_{i\geqslant 0} F^i d{*}M\Big).$$

In particular, the functor $\mathcal{R}{*}-$ is exact.

Proposition 3.25 ([Eke85, Proposition 3.3])**.** *Let M and N be left graded $\mathcal{R}$-modules. Suppose that F is bijective on N. Then the natural map $M \otimes_W N \to M{*}N$, sending $m \otimes n$ to $m{*}n$ for homogeneous elements $m \in M$ and $n \in N$, is an isomorphism.*

Proposition 3.26 ([Eke85, Lemma 4.6])**.** *Let M and N be left graded $\mathcal{R}$-modules. Then the natural map $M \otimes_W N \to M{*}N$, sending $m \otimes n$ to $m{*}n$ for homogeneous elements $m \in M$ and $n \in N$, induces an isomorphism $(\mathcal{R}_1 \otimes_{\mathcal{R}} M) \otimes_k (\mathcal{R}_1 \otimes_{\mathcal{R}} N) \xrightarrow{\cong} \mathcal{R}_1 \otimes_{\mathcal{R}} (M{*}N)$.*

Our next goal is to show that the symmetric monoidal structure on graded left $\mathcal{R}$-modules extends to the whole category $\mathcal{DG}r(\mathcal{R})$. In [Eke85, Proposition 4.5], Ekedahl obtained such a structure only at the level of the homotopy category, and one of the two inputs had to be bounded above. For our purposes, it is more convenient to construct such a symmetric monoidal structure at the ∞-categorical level and remove the bounded-above restriction.

Recall that for a collection K of simplicial sets, Lurie [Lur17, Definition 4.8.1.1] uses $\mathrm{Cat}_\infty(K)$ to denote the subcategory of Cat_∞ spanned by ∞-categories admitting colimits indexed by elements of K, and by functors preserving such colimits. Moreover, according to [Lur17, Proposition 4.8.1.3], each $\mathrm{Cat}_\infty(K)$ is a symmetric monoidal ∞-category. Furthermore, following the proof of *loc. cit.*, one sees that if $\mathcal{C}$ is a 1-category admitting colimits indexed by elements of K, then promoting $\mathcal{C}$ to a (non-unital) commutative algebra object in $\mathrm{Cat}_\infty(K)$ is equivalent to equipping $\mathcal{C}$ with a (non-unital) symmetric monoidal structure that preserves colimits indexed by elements of K in each variable.

Construction 3.27. Let K be the collection of all sets, viewed as a collection of simplicial sets via their nerves, so that colimits indexed by elements of K simply mean direct sums. By Example 3.23 and the fact that $-{*}-$ respects direct sums in one variable, we see that $M{*}N$ is a graded free left $\mathcal{R}$-module whenever both M and N are. Thus Construction 3.22 gives rise to an object $\mathcal{GM}od^{\mathrm{free}}(\mathcal{R})$ in $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Cat}_\infty(K))$ whose underlying object is the category of graded free left $\mathcal{R}$-modules. Here the superscript "nu" stands for non-unital[11] (see [Lur17, Definition 5.4.4.1]). Moreover, the functor $\mathcal{R}_1 \otimes_{\mathcal{R}} - : \mathcal{GM}od^{\mathrm{free}}(\mathcal{R}) \to \mathcal{GM}od^{\mathrm{free}}(k)$ is a morphism in $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Cat}_\infty(K))$, thanks to Proposition 3.26.

Next, we construct a symmetric monoidal structure on the connective part of $\mathcal{DG}r(\mathcal{R})$. By [Lur17, Remark 4.8.1.8], for any inclusion of collections $K \subset K'$ of simplicial sets, the functor $\mathcal{P}_K^{K'}$ of "freely adjoining colimits indexed by elements of K' while preserving those indexed by elements of K" is symmetric monoidal.

Construction 3.28. Now let K' denote the collection of all simplicial sets. Using the fact that $\mathcal{P}_K^{K'}$ is symmetric monoidal, and the non-unital commutative algebra structure on $\mathcal{GM}od^{\mathrm{free}}(\mathcal{R})$ from Construction 3.27, we see that the object $\mathcal{P}_K^{K'}(\mathcal{GM}od^{\mathrm{free}}(\mathcal{R})) \cong \mathcal{DG}r^{\leqslant 0}(\mathcal{R})$ also admits a non-unital symmetric monoidal structure. Here $\mathcal{DG}r^{\leqslant 0}(\mathcal{R})$ denotes the connective part of $\mathcal{DG}r(\mathcal{R})$.

Let us explicate the induced symmetric product in the above construction.

Remark 3.29. Recall that by Dold–Kan correspondence, each object in $\mathcal{DG}r^{\leqslant 0}(\mathcal{R})$ can be represented by a simplicial colimit of graded free left $\mathcal{R}$-modules. According to the proof of [Lur17, Proposition 4.8.1.3], the induced non-unital symmetric monoidal structure preserves arbitrary colimits in each variable and restricts to the given $-{*}-$ when both inputs are graded free. Therefore, at the homotopy level, the induced non-unital symmetric monoidal structure on $\mathcal{DG}r^{\leqslant 0}(\mathcal{R})$ is the left derived functor $-{*}^{\mathrm{L}}-$ of $-{*}-$. In particular, $\pi_0(M {*}^{\mathrm{L}} N) = \pi_0(M) {*} \pi_0(N)$.

Recall [Lur17, Theorem 5.4.4.5], which states that a non-unital symmetric monoidal structure uniquely (up to contractible choices) arises from a symmetric monoidal structure if and only if it admits a homotopy unit.

Proposition 3.30. *The non-unital symmetric monoidal structure in Construction 3.28 admits a unit; therefore it defines a symmetric monoidal structure on $\mathcal{DG}r^{\leqslant 0}(\mathcal{R})$.*

Proof. According to Remark 3.29, the symmetric product $-{*}^{\mathrm{L}}-$ is the derived functor of $-{*}-$. By Corollary 3.24, we see that one only has to resolve one of the variables by graded free modules when computing $-{*}^{\mathrm{L}}-$. Therefore, by Proposition 3.25, we see that W is a homotopy unit of $-{*}^{\mathrm{L}}-$. ☐

Remark 3.31. By the last statement of Construction 3.27, we see that $\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} (-) = \mathcal{P}_K^{K'}(\mathcal{R}_1 \otimes_{\mathcal{R}} -)$ is a symmetric monoidal functor from $\mathcal{DG}r^{\leqslant 0}(\mathcal{R})$ to $\mathcal{DG}r^{\leqslant 0}(k)$.

Lastly, we obtain an induced symmetric monoidal structure on the whole of $\mathcal{DG}r(\mathcal{R})$ by tensoring with the category of spectra Sp, as follows:

Construction 3.32. Observe that $\mathcal{DG}r^{\leqslant 0}(\mathcal{R})$ is presentable; therefore Construction 3.28 promotes it to an object of $\mathrm{CAlg}(Pr^{\mathrm{L}})$. Since the category Sp of spectra is also an object of $\mathrm{CAlg}(Pr^{\mathrm{L}})$, we obtain an object $\mathcal{DG}r^{\leqslant 0}(\mathcal{R}) \otimes \mathrm{Sp} \in \mathrm{CAlg}(Pr^{\mathrm{L}})$. According to [Lur17, Example 4.8.1.23], the underlying category is simply $\mathrm{Sp}(\mathcal{DG}r^{\leqslant 0}(\mathcal{R})) = \mathcal{DG}r(\mathcal{R})$ (see [Lur18, Remark C.1.2.10.(b)]). Therefore we obtain a symmetric monoidal structure on $\mathcal{DG}r(\mathcal{R})$. We continue to denote the induced symmetric monoidal product by $-{*}^{\mathrm{L}}-$.

Remark 3.33. Let us summarize the properties of $-{*}^{\mathrm{L}}-$ obtained in the above construction:

1. It preserves arbitrary colimits in each variable: This is a general feature of $\mathrm{CAlg}(Pr^{\mathrm{L}})$;

[11] The unit for Ekedahl's product is *not* a graded free $\mathcal{R}$-module.

2. It also preserves finite limits in each variable: This is because finite limits are always shifts of finite colimits in a stable ∞-category.

Combining these two properties, we see that for any pair of objects $M, N \in \mathcal{DG}r(\mathcal{R})$, we may compute $M *^{\mathrm{L}} N = \mathrm{colim}_{(m,n)\in\mathbb{Z}^2} \tau^{\leqslant m} M *^{\mathrm{L}} \tau^{\leqslant n} N$; and for each $\tau^{\leqslant m} M *^{\mathrm{L}} \tau^{\leqslant n} N$, we can compute by resolving one of them by graded free left $\mathcal{R}$-modules. In particular, our $- *^{\mathrm{L}} -$ extends Ekedahl's symmetric monoidal product at the homotopy level.

Remark 3.34. Taking Remark 3.31 and passing to $- \otimes \mathrm{Sp}$, we also see that $\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} -$ is a symmetric monoidal functor from $\mathcal{DG}r(\mathcal{R})$ to $\mathcal{DG}r(k)$.

Since the cohomology of de Rham–Witt complexes is always complete, it is desirable to have a symmetric monoidal structure on $\widehat{\mathcal{DG}r}(\mathcal{R})$.

Proposition 3.35. *There is a unique way to endow $\widehat{\mathcal{DG}r}(\mathcal{R})$ with a symmetric monoidal structure such that the completion functor $\widehat{(-)} \colon \mathcal{DG}r(\mathcal{R}) \to \widehat{\mathcal{DG}r}(\mathcal{R})$ is symmetric monoidal.*

We shall denote the symmetric monoidal product on $\widehat{\mathcal{DG}r}(\mathcal{R})$ by $- \widehat{*^{\mathrm{L}}} -$.

Proof. Let us mimic the proof of [Sch19, Theorem 6.2]. The requirement that $\widehat{(-)}$ be symmetric monoidal forces $- \widehat{*^{\mathrm{L}}} - := (\widehat{- *^{\mathrm{L}} -})$. Similar to loc. cit., all we need to check is that for any $M, N \in \mathcal{DG}r(\mathcal{R})$, the natural map $M *^{\mathrm{L}} N \to \widehat{M} *^{\mathrm{L}} N$ becomes an isomorphism after completion. Since both functors commute with colimits in N, we may first assume that N is bounded above (so that it is represented by a bounded above complex of graded free modules). We may then further assume that N is a graded free left $\mathcal{R}$-module concentrated in cohomological degree 0 (by considering the stupid filtration on the representing complex). In particular, at this point $- *^{\mathrm{L}} N$ is exact. Since completion is t-bounded, we can repeat the same reduction process for M. As a result, we may also assume that M is a graded free left $\mathcal{R}$-module concentrated in cohomological degree 0. Therefore, it suffices to prove that $\widehat{M * N} \to \widehat{\widehat{M} * N}$[12] is an isomorphism. Since both sides are complete and (cohomologically) bounded, by Proposition 2.37, it suffices to show that the map becomes an isomorphism after applying $\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} -$. Using Proposition 2.30, after applying $\mathcal{R}_1 \otimes^{\mathrm{L}} -$, the map becomes $\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} (M * N) \to \mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} (\widehat{M} *^{\mathrm{L}} N)$. Using Remark 3.34, the map becomes

$$(\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M) \otimes_k (\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} N) \xrightarrow{(\mathcal{R}_1 \otimes^{\mathrm{L}} \eta)\otimes \mathrm{id}} (\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} \widehat{M}) \otimes^{\mathrm{L}}_k (\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} N).$$

Using Proposition 2.30 again, we see that the map on the first factor is an isomorphism. □

Theorem 3.36 (c.f. [Eke85, Theorem II.1.1])**.** *Let X, Y be smooth varieties over k. Assume that one of the following holds:*

1. *Both X and Y are quasi-compact; or*

2. *The Hodge cohomology of either X or Y is finite-dimensional.*

Then there is a natural isomorphism:

$$R\Gamma(X, W\Omega^\bullet_{X/k}) \widehat{*^{\mathrm{L}}} R\Gamma(Y, W\Omega^\bullet_{Y/k}) \xrightarrow{\cong} R\Gamma(X \times Y, W\Omega^\bullet_{X\times Y/k}).$$

[12]Notice that we dropped the "L" for both star products, since N is graded free.

Notice that the original statement in Ekedahl's paper seems to be incorrect: He did not assume any finiteness condition, with the simplest counterexample being taking both $X = Y = \bigsqcup_{\mathbb{Z}} \operatorname{Spec}(k)$. However, with the technical assumption added above, Ekedahl's proof holds. We shall prove the above statement in greater generality, allowing X and Y to be smooth algebraic stacks (Theorem 4.16), so let us postpone its proof until there.

The remainder of this subsection is devoted to some concrete computations that will be useful for later purposes.

Notation 3.37. We write $\mathbb{D}(\alpha_p)$ for the graded left $\mathcal{R}$-module given by k placed in degree 0, on which the operators F, V, and d act trivially. For coprime integers $i \geqslant 1, j \geqslant 0$, we denote by $E_{j/i+j}$ the graded left $\mathcal{R}$-module $E_{j/i+j} := \mathcal{R}^0/\mathcal{R}^0(F^i - V^j)$, concentrated in degree 0. Equivalently, $E_{j/i+j}$ is the free W-module generated by

$$\{f_i, f_{i-1}, \dots, f_1, e_0, e_1, \dots, e_{j-1}\},$$

with $d = 0$, and the semi-linear operators F and V given by

$$\begin{aligned} F f_s &= f_{s+1} \quad (0 \le s \le i-1), & F e_t &= p\, e_{t-1} \quad (1 \le t \le j), \\ V f_s &= p\, f_{s-1} \quad (1 \le s \le i), & V e_t &= e_{t+1} \quad (0 \le t \le j-1). \end{aligned}$$

Here we use the conventions $f_0 = e_0$ and $e_j = f_i$.

Observe that F commutes with $F^i - V^j$, so right multiplication by F on $\mathcal{R}^0$ descends to a well-defined map of graded left $\mathcal{R}$-modules: $E_{j/i+j} \xrightarrow{\cdot F} E_{j/i+j}$. For $E_{1/2}$, one verifies that this map fits into a short exact sequence $0 \longrightarrow E_{1/2} \xrightarrow{\cdot F} E_{1/2} \longrightarrow \mathbb{D}(\alpha_p) \longrightarrow 0$.

Lemma 3.38. *There is a natural decomposition*

$$\widehat{\mathcal{R}} \widehat{*^{\mathrm{L}}} \mathbb{D}(\alpha_p) \cong \Big(\prod_{i>0} V^i(1 * \mathbb{D}(\alpha_p))\Big) \oplus \Big(\bigoplus_{i \geqslant 0} F^i * \mathbb{D}(\alpha_p)\Big) \oplus \Big(\prod_{i>0} dV^i(1 * \mathbb{D}(\alpha_p))\Big) \oplus \Big(\bigoplus_{i \geqslant 0} F^i d * \mathbb{D}(\alpha_p)\Big)$$

compatible with the decomposition in Corollary 3.24.[13] *In particular, the object* $\widehat{\mathcal{R}} \widehat{*^{\mathrm{L}}} \mathbb{D}(\alpha_p) \in \widehat{\mathcal{DG}r}(\mathcal{R})$ *is concentrated in cohomological degree* 0.

Proof. By Proposition 3.35, we need to apply the completion functor $\widehat{(-)}$ to the graded left $\mathcal{R}$-module $\mathcal{R} * \mathbb{D}(\alpha_p)$. In the decomposition displayed in Corollary 3.24, we note that the action of V annihilates the latter three summands and that the action of p is 0. Therefore, by the resolution of the pro-system $\{\mathcal{R}_n\}$ in Proposition 2.27, we obtain an isomorphism of pro-systems:

$$\{\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} (\mathcal{R} * \mathbb{D}(\alpha_p))\} \cong \left\{ \Big(\bigoplus_{0<i<n} V^i(1 * \mathbb{D}(\alpha_p))\Big) \oplus \Big(\bigoplus_{i \geqslant 0} F^i * \mathbb{D}(\alpha_p)\Big) \oplus \Big(\bigoplus_{0<i<n} dV^i(1 * \mathbb{D}(\alpha_p))\Big) \oplus \Big(\bigoplus_{i \geqslant 0} F^i d * \mathbb{D}(\alpha_p)\Big) \right\}.$$

Taking the inverse limit gives our result. □

Proposition 3.39. *The object* $E_{1/2} \widehat{*^{\mathrm{L}}} \mathbb{D}(\alpha_p)$ *is cohomologically concentrated in* $[-1, 0]$. *Its cohomologies are given by the following dominoes:* $\mathrm{H}^{-1} = U_{-1}$ *and* $\mathrm{H}^0 = U_1$.

[13] This compatibility, together with the requirement that the action of F, V, and d be continuous with respect to the topology from Definition 2.14, automatically determines the graded left $\mathcal{R}$-module structure on the right-hand side.

Proof. By [Eke85, Lemma III.1.5], we have the following resolution of graded left $\mathcal{R}$-modules:

$$0 \to \widehat{\mathcal{R}} \xrightarrow{\cdot(F^i - V^j)} \widehat{\mathcal{R}} \to E_{j/i+j} \to 0.$$

Therefore, we need to consider the map $\widehat{\mathcal{R}}\widehat{*^{\mathrm{L}}}\mathbb{D}(\alpha_p) \to \widehat{\mathcal{R}}\widehat{*^{\mathrm{L}}}\mathbb{D}(\alpha_p)$ given by applying $\widehat{(-)}$ to

$$\mathcal{R}*\mathbb{D}(\alpha_p) \xrightarrow{(-\cdot(F-V))*\mathrm{id}} \mathcal{R}*\mathbb{D}(\alpha_p).$$

Using the decomposition in Lemma 3.38, we obtain the following descriptions of the kernel and cokernel:

- The kernel is given by $\prod_{i>0} V^i(1*\mathbb{D}(\alpha_p))$ in grading 0 and $\prod_{i\geqslant 0} dV^i(1*\mathbb{D}(\alpha_p))$ in grading 1;
- The cokernel is given by $\prod_{i\geqslant 0} V^i(1*\mathbb{D}(\alpha_p))$ in grading 0 and $\prod_{i>0} dV^i(1*\mathbb{D}(\alpha_p))$ in grading 1.

This completes our computation. ☐

3.3 Diagonal t-structure

Recall that in [Eke86], Ekedahl defined a t-structure on $\mathcal{DG}r_c^b(\mathcal{R})$, which he termed the "diagonal t-structure", and proved his famous inequality Theorem 3.49. The goal of this subsection is to extend his diagonal t-structure to the whole $\mathcal{DG}r_c(\mathcal{R})$. We begin by reviewing his definition of the diagonal t-structure on $\mathcal{DG}r_c^b(\mathcal{R})$.

Definition 3.40. For each coherent graded left $\mathcal{R}$-module M, define $M^{\leqslant i}$ and $M^{\geqslant i}$ to be the coherent $\mathcal{R}$-modules given by

$$M^{\leqslant i} := (\cdots \xrightarrow{d} M^{i-1} \xrightarrow{d} M^i \xrightarrow{d} F^\infty d(M^i) \to 0 \to \cdots),$$
$$M^{\geqslant i} := (\cdots \to 0 \to M^i/F^\infty d(M^{i-1}) \xrightarrow{d} M^{i+1} \to \cdots).$$

We denote

$$M^{[i,i]} := (M^{\leqslant i})^{\geqslant i} = (M^{\geqslant i})^{\leqslant i} = (\cdots \to 0 \to M^i/F^\infty d(M^{i-1}) \xrightarrow{d} F^\infty d(M^i) \to 0 \to \cdots).$$

Note that these two formulas define functors $(-)^{\leqslant i}$ and $(-)^{\geqslant i}$. Moreover, there are natural transformations $(-)^{\leqslant i} \to \mathrm{id} \to (-)^{\geqslant j}$ for each i and j. The connective (resp. co-connective) part of Ekedahl's diagonal t-structure on $\mathcal{DG}r_c^b(\mathcal{R})$ is then defined by

$$\widetilde{\mathcal{DG}r}_c^{b,\leqslant 0} := \{M \in \mathcal{DG}r_c^b(\mathcal{R}) \mid \mathrm{H}^i(M)^{\leqslant -i} \xrightarrow{\cong} \mathrm{H}^i(M)\},$$
$$\widetilde{\mathcal{DG}r}_c^{b,\geqslant 0} := \{M \in \mathcal{DG}r_c^b(\mathcal{R}) \mid \mathrm{H}^i(M) \xrightarrow{\cong} \mathrm{H}^i(M)^{\geqslant -i}\}.$$

Theorem 3.41 ([Eke86, Chapter 0 & I])**.** *The above defines a t-structure on $\mathcal{DG}r_c^b(\mathcal{R})$. Moreover, if we denote the canonical connective/co-connective truncation functors by $\tau^{\leqslant 0}$ and $\tau^{\geqslant 0}$, and the truncation functors with respect to the diagonal t-structure by $\widetilde{\tau}^{\leqslant 0}$ and $\widetilde{\tau}^{\geqslant 0}$, then they satisfy the following properties:*

1. *For each pair of integers (i, j) and each object $M \in \mathcal{DG}r_c^b(\mathcal{R})$, the map $\mathrm{H}^i(\widetilde{\tau}^{\leqslant j}(M)) \to \mathrm{H}^i(M)$ induces a natural identification $\mathrm{H}^i(\widetilde{\tau}^{\leqslant j}(M)) \xrightarrow{\cong} \mathrm{H}^i(M)^{\leqslant j-i}$; similarly the map $\mathrm{H}^i(M) \to \mathrm{H}^i(\widetilde{\tau}^{\geqslant j}(M))$ induces a natural identification $\mathrm{H}^i(M)^{\geqslant j-i} \xrightarrow{\cong} \mathrm{H}^i(\widetilde{\tau}^{\geqslant j}(M))$.*
2. *The canonical truncation functors preserve connective/co-connective part of the diagonal t-structure; similarly the diagonal truncation functors preserve canonical connective/co-connective part.*

3. *For each pair of integers (i, j), we have the following natural transformations which are all equivalences: $\widetilde{\tau}^{\leqslant j} \circ \tau^{\leqslant i} \xrightarrow{\cong} \tau^{\leqslant i} \circ \widetilde{\tau}^{\leqslant j}$, $\tau^{\geqslant i} \circ \widetilde{\tau}^{\leqslant j} \xrightarrow{\cong} \widetilde{\tau}^{\leqslant j} \circ \tau^{\geqslant i}$, $\widetilde{\tau}^{\geqslant i} \circ \tau^{\leqslant j} \xrightarrow{\cong} \tau^{\leqslant j} \circ \widetilde{\tau}^{\geqslant i}$, and $\tau^{\geqslant i} \circ \widetilde{\tau}^{\geqslant j} \xrightarrow{\cong} \widetilde{\tau}^{\geqslant j} \circ \tau^{\geqslant i}$.*

Let us explain how the natural transformation $\tau^{\geqslant i} \circ \widetilde{\tau}^{\leqslant j} \xrightarrow{\cong} \widetilde{\tau}^{\leqslant j} \circ \tau^{\geqslant i}$ is defined. Applying $\widetilde{\tau}^{\leqslant j}$ to the natural arrow $\mathrm{id} \to \tau^{\geqslant i}$ gives an arrow $\widetilde{\tau}^{\leqslant j} \to \widetilde{\tau}^{\leqslant j} \circ \tau^{\geqslant i}$. By the second property above, we see that the target lies in $\mathcal{DG}r_c^{b,\geqslant i}(\mathcal{R})$. Hence the above arrow factors through $\tau^{\geqslant i} \circ \widetilde{\tau}^{\leqslant j}$. The other arrows are defined similarly.[14]

For the reader's convenience, we include a sketch of Ekedahl's proof.

Proof sketch. We say that a coherent $\mathcal{R}$-module M is "concentrated in grading m" if $M^\ell = 0$ for all $\ell \notin \{m, m+1\}$, and $M^{m+1} = F^\infty d(M^m)$. Using the resolutions in [Eke85, Lemma III.1.5], Ekedahl observed ([Eke86, Proposition 0.4.1]) that if M and N are coherent $\mathcal{R}$-modules concentrated in gradings m and n, respectively, then $\mathrm{Ext}^i_{\mathcal{R}}(M, N) = 0$ whenever $i < (n - m)$ and $2 \leqslant i \leqslant (n - m)$. The first vanishing result immediately implies that if $M \in \widetilde{\mathcal{DG}r}_c^{b,\leqslant 0}(\mathcal{R})$ and $N \in \widetilde{\mathcal{DG}r}_c^{b,\geqslant 1}(\mathcal{R})$, then $\mathrm{Hom}_{\mathcal{DG}r(\mathcal{R})}(M, N) = 0$.

Next, we show the existence of a diagonal connective cover that moreover satisfies property (1). To this end, we argue by induction on the cohomological amplitude of $M \in \mathcal{DG}r_c^b(\mathcal{R})$. If the cohomological amplitude is 0, that is, if M is a cohomological shift of a coherent $\mathcal{R}$-module, the existence of a diagonal connective cover satisfying property (1) is obvious.

In general, assume that the statement is known for all objects of cohomological amplitude $\leqslant (n-1)$. Suppose that $M \in \mathcal{DG}r_c^{[a-n,a]}(\mathcal{R})$, and consider the exact triangle $\tau^{<a}M \to M \to \mathrm{H}^a(M)[-a]$. By the induction hypothesis, both $\tau^{<a}M$ and $\mathrm{H}^a(M)[-a]$ admit diagonal connective covers satisfying property (1). The second vanishing in the first paragraph then implies that the composite

$$\widetilde{\tau}^{\leqslant 0}(\mathrm{H}^a(M)[-a]) \to \mathrm{H}^a(M)[-a] \to (\tau^{<a}M)[1] \to \widetilde{\tau}^{\geqslant 1}(\tau^{<a}M)[1]$$

is 0. Therefore, this composite canonically factors through $\widetilde{\tau}^{\leqslant 0}(\mathrm{H}^a(M)[-a]) \to \widetilde{\tau}^{\leqslant 0}(\tau^{<a}M)[1]$. Taking its fiber, which then necessarily maps to M, produces the desired diagonal connective cover of M satisfying property (1).

Finally, properties (2) and (3) are formal consequences of property (1). □

We now extend Ekedahl's diagonal t-structure to $\mathcal{DG}r_c(\mathcal{R})$ without the cohomological boundedness constraint.

Definition 3.42. We define the following full subcategories of $\mathcal{DG}r_c(\mathcal{R})$:

$$\widetilde{\mathcal{DG}r}_c^{\leqslant 0} \coloneqq \{M \in \mathcal{DG}r_c(\mathcal{R}) \mid \mathrm{H}^i(M)^{\leqslant -i} \xrightarrow{\cong} \mathrm{H}^i(M)\},$$
$$\widetilde{\mathcal{DG}r}_c^{\geqslant 0} \coloneqq \{M \in \mathcal{DG}r_c(\mathcal{R}) \mid \mathrm{H}^i(M) \xrightarrow{\cong} \mathrm{H}^i(M)^{\geqslant -i}\}.$$

Theorem 3.43. *The above defines a t-structure on $\mathcal{DG}r_c(\mathcal{R})$. Moreover, if we denote the canonical connective and co-connective truncation functors by $\tau^{\leqslant 0}$ and $\tau^{\geqslant 0}$, and the truncation functors with respect to the diagonal t-structure by $\widetilde{\tau}^{\leqslant 0}$ and $\widetilde{\tau}^{\geqslant 0}$, then they satisfy the following properties:*

1. *For each pair of integers (i, j) and each object $M \in \mathcal{DG}r_c(\mathcal{R})$, the map $\mathrm{H}^i(\widetilde{\tau}^{\leqslant j}(M)) \to \mathrm{H}^i(M)$ induces a natural identification $\mathrm{H}^i(\widetilde{\tau}^{\leqslant j}(M)) \xrightarrow{\cong} \mathrm{H}^i(M)^{\leqslant j-i}$. Similarly, the map $\mathrm{H}^i(M) \to \mathrm{H}^i(\widetilde{\tau}^{\geqslant j}(M))$*

[14] Namely, we always apply diagonal truncations to the natural transformation between id and the canonical truncations, and then obtain the desired arrow by observing that the source or the target lies in the appropriate shifts of the connective or co-connective part of the canonical t-structure, by the second property above.

induces a natural identification $\mathrm{H}^i(M)^{\geqslant j-i} \xrightarrow{\cong} \mathrm{H}^i(\widetilde{\tau}^{\geqslant j}(M))$.

2. *The canonical truncation functors preserve the connective and co-connective parts of the diagonal t-structure. Similarly, the diagonal truncation functors preserve the connective and co-connective parts of the canonical t-structure.*

3. *For each pair of integers (i,j), we have the following natural transformations, all of which are equivalences:*

$$\widetilde{\tau}^{\leqslant j} \circ \tau^{\leqslant i} \xrightarrow{\cong} \tau^{\leqslant i} \circ \widetilde{\tau}^{\leqslant j}, \quad \tau^{\geqslant i} \circ \widetilde{\tau}^{\leqslant j} \xrightarrow{\cong} \widetilde{\tau}^{\leqslant j} \circ \tau^{\geqslant i},$$
$$\widetilde{\tau}^{\geqslant i} \circ \tau^{\leqslant j} \xrightarrow{\cong} \tau^{\leqslant j} \circ \widetilde{\tau}^{\geqslant i}, \quad \tau^{\geqslant i} \circ \widetilde{\tau}^{\geqslant j} \xrightarrow{\cong} \widetilde{\tau}^{\geqslant j} \circ \tau^{\geqslant i}.$$

Proof. Suppose that $M \in \widetilde{\mathcal{DG}r}_c^{\leqslant 0}(\mathcal{R})$ and $N \in \widetilde{\mathcal{DG}r}_c^{\geqslant 1}(\mathcal{R})$. We need to show that $\mathrm{Hom}_{\mathcal{DG}r(\mathcal{R})}(M,N) = 0$. Using the Postnikov filtrations on M and N, we have a canonical isomorphism

$$\mathrm{RHom}_{\mathcal{DG}r(\mathcal{R})}(M,N) \cong \mathrm{Rlim}_{n\to\infty} \mathrm{RHom}_{\mathcal{DG}r(\mathcal{R})}(\tau^{[-n,n]}M, \tau^{[-n,n]}N),$$

which has no $\mathrm{H}^{\leqslant 0}$. Indeed, for each n, we have $\tau^{[-n,n]}M \in \widetilde{\mathcal{DG}r}_c^{b,\leqslant 0}(\mathcal{R})$, and $\tau^{[-n,n]}N \in \widetilde{\mathcal{DG}r}_c^{b,\geqslant 1}(\mathcal{R})$, so $\mathrm{RHom}_{\mathcal{DG}r(\mathcal{R})}(\tau^{[-n,n]}M, \tau^{[-n,n]}N)$ has no $\mathrm{H}^{\leqslant 0}$ by Theorem 3.41.

Next, we show the existence of diagonal truncations. Let $M \in \mathcal{DG}r_c(\mathcal{R})$ and consider the diagonal connective covers of its Postnikov truncations $N^{[a,b]} \coloneqq \widetilde{\tau}^{\leqslant 0}(\tau^{[a,b]}M)$, where $a \leqslant b$ runs through all such pairs of integers. By property (2) of Theorem 3.41, we see that $N^{[a,b]} \in \mathcal{DG}r_c^{[a,b]}(\mathcal{R})$. Moreover, by property (3) of Theorem 3.41, for each pair of integers $a \leqslant b$, the natural maps

$$N^{[a,b]} \longrightarrow \tau^{[a,b]}N^{[a,b+1]},$$
$$\tau^{[a,b]}N^{[a-1,b]} \longrightarrow N^{[a,b]}$$

are isomorphisms. Therefore, we see that $N \coloneqq \lim_{a\to-\infty} \mathrm{colim}_{b\to\infty} N^{[a,b]}$ exists together with natural isomorphisms $\tau^{[a,b]}N \cong N^{[a,b]}$ for all pairs of integers $a \leqslant b$.

Using property (3) of Theorem 3.41 again, each $N^{[a,b]}$ has a natural map to $\tau^{[a,b]}M$ compatible with Postnikov truncations. Therefore, we obtain a natural map $N \to M$ whose induced map on H^i coincides with the map induced by $N^{[a,b]} \to \tau^{[a,b]}M$ for any $a \leqslant b$ with $i \in [a,b]$. Hence property (1) of Theorem 3.41 implies that the map $N \to M$ satisfies the first half of property (1) here.

Therefore, we see that $N \in \widetilde{\mathcal{DG}r}_c^{\leqslant 0}(\mathcal{R})$ and $\mathrm{Cone}(N \to M) \in \widetilde{\mathcal{DG}r}_c^{\geqslant 1}(\mathcal{R})$. Moreover, the second half of property (1) also follows by considering the long exact sequence associated with $N \to M \to \mathrm{Cone}(N \to M)$.

Finally, properties (2)–(3) follow formally from property (1). ☐

Notation 3.44. We denote the heart of the diagonal t-structure by $\mathbf{\Delta}' \coloneqq \widetilde{\mathcal{DG}r}_c^{\leqslant 0}(\mathcal{R}) \cap \widetilde{\mathcal{DG}r}_c^{\geqslant 0}(\mathcal{R})$. We denote the associated cohomology functors by $\widetilde{\mathrm{H}}^n(\text{-}) \coloneqq (\widetilde{\tau}^{\leqslant n} \circ \widetilde{\tau}^{\geqslant n}(\text{-}))[n] \colon \mathcal{DG}r_c(\mathcal{R}) \to \mathbf{\Delta}'$.

Lemma 3.45. *Let $M \in \mathcal{DG}r_c(\mathcal{R})$, for each pair of integers (i,j) we have* $\mathrm{H}^j(\widetilde{\mathrm{H}}^i(M)) \cong \mathrm{H}^{i+j}(M)^{[-j,-j]}$.

Proof. Let us compute:

$$\mathrm{H}^j(\widetilde{\mathrm{H}}^i(M)) \coloneqq \mathrm{H}^j\Big((\widetilde{\tau}^{\leqslant i} \circ \widetilde{\tau}^{\geqslant i})(M)[i]\Big) = \mathrm{H}^{i+j}\Big(\widetilde{\tau}^{\leqslant i}(\widetilde{\tau}^{\geqslant i}(M))\Big) \cong \mathrm{H}^{i+j}(\widetilde{\tau}^{\geqslant i}(M))^{\leqslant -j} \cong \mathrm{H}^{i+j}(M)^{[-j,-j]}.$$

In the last two identifications, we have used property (1) of Theorem 3.43. ☐

Remark 3.46. In particular, as a consequence of Lemma 3.45, for any smooth proper variety X over k of dimension $\leqslant D$, we have $\mathrm{R}\Gamma(X, W\Omega^{\bullet}_{X/k})$ lives in $\widetilde{\mathcal{DG}r}_c^{[0,2D]}(\mathcal{R})$.

Lemma 3.47. *Let $M \in \mathbf{\Delta}' \subset \mathcal{DG}r_c(\mathcal{R})$, the following are equivalent:*

1. *$M \in \mathbf{\Delta} \subset \mathbf{\Delta}'$;*
2. *M has bounded cohomology;*
3. *M has bounded grading.*

Proof. The equivalence between (1) and (2) follows from the fact that $\mathbf{\Delta} = \mathbf{\Delta}' \cap \mathcal{DG}r_c^b(\mathcal{R})$. The equivalence between (2) and (3) follows from the definition of $M \in \mathbf{\Delta}'$: this implies that the i-th cohomology of M has no graded pieces other than those in gradings $-i$ and $-i+1$. □

Lemma 3.48. *Let $M \in \mathcal{DG}r_c(\mathcal{R})$ be an object that is cohomologically bounded below and grading-left bounded (that is, the grading $\leqslant -N$ pieces of M vanish for some integer N). Then for each $i \in \mathbb{Z}$, we have $\widetilde{\mathrm{H}}^i(M) \in \mathbf{\Delta} \subset \mathbf{\Delta}'$.*

Proof. Fix an integer i. By Lemma 3.47, it suffices to show that $\mathrm{H}^j(\widetilde{\mathrm{H}}^i(M)) = 0$ whenever j is sufficiently large or sufficiently small. Using Lemma 3.45, we obtain the following:

1. If j is sufficiently small, the vanishing follows from the assumption that M is cohomologically bounded below.
2. If j is sufficiently large, the vanishing follows from the assumption that M is grading-left bounded.

This completes the proof. □

Using the diagonal t-structure on $\mathcal{DG}r_c^b(\mathcal{R})$, Ekedahl established a fundamental inequality that reveals deep connections between Hodge–Witt numbers and Hodge numbers.

Theorem 3.49 (Ekedahl's inequality [Eke86, Theorem IV.3.3])**.** *Let $M \in \mathcal{DG}r_c^b(\mathcal{R})$. Then for any $(i,j) \in \mathbb{Z}^2$, we have $h_W^{i,j}(M) \leqslant h^{i,j}(M)$. Consequently, if X is a smooth proper variety over a perfect field k of characteristic p, then for any $(i,j) \in \mathbb{N}^2$ we have $h_W^{i,j}(X) \leqslant h^{i,j}(X)$.*

This result provides the following insight into the behaviour of the Hodge–Witt numbers $h_W^{i,j}$.

Proposition 3.50 ([Eke86, Proposition III.4.1], [Eke86, Theorem IV.1.2(i)])**.** *Let $M \in \mathcal{DG}r_c^b(\mathcal{R})$. The following conditions are equivalent:*

(1) *The cohomology groups of the total complex $\mathrm{Tot}(M) \in \mathcal{D}(W)$ are torsion-free, and the Hodge–de Rham spectral sequence (see Construction 2.36)*

$$E_1^{i,j} = \mathrm{H}^j(\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} M)^i \;\Rightarrow\; \mathrm{H}^{i+j}(k \otimes^{\mathrm{L}}_W \mathrm{Tot}(M))$$

degenerates at the E_1-page.

(2) *For every (i,j), we have $h^{i,j} = h_W^{i,j}$.*

(3) *For each $n \in \mathbb{Z}$, let b_n denote the dimension of $\mathrm{H}^n(\mathrm{Tot}(M))[1/p]$. Then $\sum_{i+j=n} h^{i,j} = b_n$.*

Proof. A direct computation shows that $\sum_{i+j=n} h_W^{i,j}(M) = \sum_{i+j=n} m^{i,j}(M) = b_n$. Therefore, by Ekedahl's inequality Theorem 3.49, condition (2) is equivalent to condition (3).

By the universal coefficient theorem and the spectral sequence of Construction 2.36, we have

$$b_n \leqslant \dim_k \mathrm{H}^n(k \otimes_W^{\mathrm{L}} \mathrm{Tot}(M)) \leqslant \sum_{i+j=n} h^{i,j}.$$

Condition (1) is precisely the statement that both inequalities are equalities for all $n \in \mathbb{Z}$. Thus condition (1) is also equivalent to condition (3). □

Definition 3.51 ([Eke86, Definition IV.1.1]). An object $M \in \mathcal{DG}r_c^b(\mathcal{R})$ is called *Mazur–Ogus* if it satisfies the equivalent conditions of Proposition 3.50. A smooth proper variety X over a perfect field k of characteristic p is called *Mazur–Ogus* if the object $\mathrm{R}\Gamma(X, W\Omega^\bullet_{X/k}) \in \mathcal{DG}r_c^b(\mathcal{R})$ is Mazur–Ogus.

Remark 3.52. In Ekedahl's original definition [Eke86, Definition I.4.3], the Mazur–Ogus condition for an object $M \in \mathbf{\Delta}$ is formulated differently. However, he later showed in [Eke86, Theorem IV.1.2(i)] that the two definitions agree.

Theorem 3.53 ([Eke86, Theorem IV.1.2(iv)]). *Assume that $M \in \mathcal{DG}r_c^b(\mathcal{R})$ is Mazur–Ogus in the sense of Definition 3.51. Then there is an isomorphism in $\mathcal{DG}r_c^b(\mathcal{R})$:*

$$M \simeq \bigoplus_n \widetilde{\mathrm{H}}^n(M)[-n].$$

Proposition 3.54 ([Eke86, Proposition III.4.11]). *Assume $M, N \in \mathbf{\Delta}$ are Mazur–Ogus objects. Then $M \widehat{\ast^{\mathrm{L}}} N \in \mathbf{\Delta}$ is also Mazur–Ogus. Consequently, for two Mazur–Ogus objects $M, N \in \mathcal{DG}r_c^b(\mathcal{R})$, we have*

1. *$M \widehat{\ast^{\mathrm{L}}} N$ is Mazur–Ogus; and*
2. *$\widetilde{\mathrm{H}}^n(M \widehat{\ast^{\mathrm{L}}} N) \cong \bigoplus_{i+j=n} \widetilde{\mathrm{H}}^i(M) \widehat{\ast^{\mathrm{L}}} \widetilde{\mathrm{H}}^j(N)$.*

Example 3.55. Let $\mathbb{P}^n$ be the projective space over k of dimension n. For each $0 \leq i \leq n$, the crystalline cohomology satisfies $\mathrm{H}^{2i}_{\mathrm{crys}}(\mathbb{P}^n/W) \cong W$, where the Frobenius operator acts as p^i times the usual Witt-vector Frobenius, and $\mathrm{H}^{2i+1}_{\mathrm{crys}}(\mathbb{P}^n/W) = 0$. The Hodge numbers are given by $h^{i,i}(\mathbb{P}^n) = 1$ and $h^{m,n}(\mathbb{P}^n) = 0$ for $m \neq n$.

A direct computation shows that the equality $\sum_{i+j=n} h^{i,j} = b_n$ holds for all n. Hence $\mathbb{P}^n$ is Mazur–Ogus in the sense of Definition 3.51. Unwinding the equality $h_W^{i,j} = h^{i,j}$, together with the description of the slopes of its crystalline cohomology, we see that $\mathbb{P}^n$ is Hodge–Witt in the sense of Definition 2.43. By Theorem 2.44, we obtain a canonical decomposition

$$(\mathrm{H}^m_{\mathrm{crys}}(\mathbb{P}^n/W), \varphi_{\mathrm{crys}}) \cong \bigoplus_{i+j=m} (\mathrm{H}^j(\mathbb{P}^n, W\Omega^i_{\mathbb{P}^n/k}), p^i F).$$

Consequently, there is a decomposition in $\mathcal{DG}r_c^b(\mathcal{R})$:

$$\mathrm{R}\Gamma(\mathbb{P}^n, W\Omega^\bullet_{\mathbb{P}^n/k}) \;\cong\; \bigoplus_{0 \leq i \leq n} W(-i)[-i].$$

For any Dieudonné module (M, F, V), there is a canonical decomposition $(M, F, V) = (M_{\mathrm{uni}}, F_{\mathrm{uni}}, V_{\mathrm{uni}}) \oplus (M_{\mathrm{mul}}, F_{\mathrm{mul}}, V_{\mathrm{mul}})$, where M_{uni} is the summand on which V acts topologically nilpotently, and M_{mul} is the summand on which V is bijective. We introduce the following notation.

Notation 3.56. For any multiplicative-type Dieudonné module (M, F, V) (that is, V is bijective), we denote by M' the Dieudonné module (M, V^{-1}, pV).

Lemma 3.57. *The endofunctor on the category of finite-type Dieudonné modules given by $(M, F, V) \longmapsto (M_{\mathrm{mul}})'$ is exact.*

Proof. Taking the direct summand on which V is bijective is an exact operation. Moreover, the passage from (M, F, V) (with bijective V) to (M, V^{-1}, pV) is also exact. □

Lemma 3.58. *For any abelian variety A, there is a canonical isomorphism*

$$\widetilde{\mathrm{H}}^1(W\Omega^\bullet(A)) \cong \mathrm{H}^1_{\mathrm{crys}}(A/W)_{\mathrm{uni}} \oplus (\mathrm{H}^1_{\mathrm{crys}}(A/W)_{\mathrm{mul}})'(-1)[1],$$

where $\mathbb{D}(-)$ denotes the usual contravariant Dieudonné module functor.

Proof. By [Ill79, Corollaire II.2.17, Proposition II.2.19], any smooth proper variety satisfies the condition of [IR83, IV.2.15.6] for $n = 1$. Using [IR83, Théorème IV.4.5], we obtain a canonical decomposition

$$(\mathrm{H}^1_{\mathrm{crys}}(A/W), \varphi_{\mathrm{crys}}) \cong (\mathrm{H}^1(WO_A), F) \oplus (\mathrm{H}^0(W\Omega^1_{A/k}), pF).$$

Consequently, there are canonical isomorphisms of Dieudonné modules

$$\mathrm{H}^1(WO_A) \cong \mathrm{H}^1_{\mathrm{crys}}(A/W)_{\mathrm{uni}}, \qquad \mathrm{H}^0(W\Omega^1_{A/k}) \cong (\mathrm{H}^1_{\mathrm{crys}}(A/W)_{\mathrm{mul}})'.$$

By Lemma 3.45, the object $\widetilde{\mathrm{H}}^1(W\Omega^\bullet(A))$ is an extension of $\mathrm{H}^1(WO_A)$ by $\mathrm{H}^0(W\Omega^1_{A/k})(-1)[1]$. Since these two terms are concentrated in different gradings, the extension is canonically split. Therefore we obtain the desired canonical isomorphism. □

4 de Rham–Witt cohomology of smooth stacks

Let k be a perfect field of characteristic $p > 0$. We begin by following [ABM21, Section 2] to extend the definition of the de Rham–Witt complex (and its cohomology) from smooth k-schemes to smooth geometric Artin stacks over k. For a brief review of geometric Artin stacks, we refer the reader to [KP24, Appendix A.1], especially [KP24, Theorem A.1.6] and the discussion preceding it.

Notation 4.1 (cf. [ABM21, Notation 2.1])**.** Let Sm_k denote the category of smooth k-algebras. Its opposite category $\mathrm{Sm}_k^{\mathrm{op}}$, equipped with the Grothendieck topology in which covers are given by finite families of smooth morphisms that are jointly surjective, is called the *smooth site* of k. A geometric Artin stack X/k is said to be *smooth* if there exists a smooth cover $U \twoheadrightarrow X$ with U a smooth k-scheme. This notion agrees with the smoothness of the morphism $X \to \mathrm{Spec}(k)$ as defined in [KP24, A.1.15–16]. We denote by SmStk_k the category of smooth geometric Artin stacks over k, again equipped with the smooth topology.

Proposition 4.2. *The $\mathcal{DG}r(\mathcal{R})$-valued functor $A \mapsto W\Omega^\bullet(A/k)$ defines a sheaf on $\mathrm{Sm}_k^{\mathrm{op}}$. Similarly, for each n, the $\mathcal{DG}r(W_n[d]/d^2)$-valued functor $A \mapsto W_n\Omega^\bullet_{A/k}$ defines a sheaf on $\mathrm{Sm}_k^{\mathrm{op}}$.*

Proof. By [Sta26, Tag 055V], it suffices to show that the said functors satisfy the sheaf property with respect to étale coverings, which is shown in [Ill79, Proposition II.1.13, II.1.14]. □

Remark 4.3. Let us mention that a more general descent statement for the de Rham–Witt complex is known, provided that one works in the animated setting, see [DM25, Proposition 2.19].

Construction 4.4 (de Rham–Witt cohomology of geometric Artin stacks)**.** Let X/k be a smooth geometric Artin stack. We set

$$W\Omega^\bullet(X/k) \coloneqq \mathrm{R}\Gamma(\mathrm{Sm}^{\mathrm{op}}_{k,/X}, W\Omega^\bullet) \in \mathcal{DG}r(\mathcal{R}).$$

By Proposition 4.2, we see that the assignment $X/k \mapsto W\Omega^\bullet(X/k)$ is again a sheaf on SmStk_k. In particular, one may "compute" the value $W\Omega^\bullet(X/k)$ by induction on the geometricity of X. When X is (-1)-geometric, which means it is a smooth affine scheme $X = \mathrm{Spec}(A)$, it is given by the usual $W\Omega^\bullet(A/k)$. Similarly, if X is a smooth scheme, then we can compute $W\Omega^\bullet(X/k)$ by Zariski descent. In general, suppose X is a smooth n-geometric stack. Then there exists an $(n-1)$-representable smooth surjection $\pi \colon U \twoheadrightarrow X$ from a smooth scheme U. All the terms $U^{(*)} \coloneqq U^{\times_X (*+1)}$ in the Čech nerve of π are smooth $(n-1)$-geometric stacks, hence their de Rham–Witt cohomologies are "computed" by induction. Moreover, we have an equivalence in $\mathcal{DG}r(\mathcal{R})$:

$$W\Omega^\bullet(X/k) \xrightarrow{\cong} \mathrm{Tot}\big(\ W\Omega^\bullet(U^{(0)}/k) \rightrightarrows W\Omega^\bullet(U^{(1)}/k) \rightrightarrows W\Omega^\bullet(U^{(2)}/k) \rightrightarrows \cdots\ \big).$$

Let us establish some easy properties of the above construction. We say that an object in $\mathcal{DG}r(\mathcal{R})$ is grading-left bounded by 0 if it has no negative grading pieces.

Lemma 4.5. *Let X/k be a smooth geometric Artin stack. Then $W\Omega^\bullet(X/k) \in \widehat{\mathcal{DG}r}(\mathcal{R})$ and it is grading-left bounded by* 0.

Proof. When X is an affine scheme, the completeness of its de Rham–Witt complex follows from Theorem 2.34. The bound on the grading follows from the definition. The completeness for general X follows from Proposition 2.29, whereas the bound on the grading follows from the fact that limits are computed "grading-wise". ∎

Remark 4.6. For each smooth algebra A/k, the multiplication on $W\Omega^\bullet_{A/k}$ satisfies the properties listed in Construction 3.22 (see [Ill79, Théorème I.1.3, Proposition I.2.18]), making it an associative algebra in the symmetric monoidal category of graded left $\mathcal{R}$-modules. Since $W\Omega^\bullet_{A/k}$ is concentrated in cohomological degree 0, and $-*^{\mathrm{L}}-$ preserves the connective part, we see that $W\Omega^\bullet_{-/k}$ defines a sheaf on $\mathrm{Sm}^{\mathrm{op}}_k$ valued in $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{DG}r(\mathcal{R}))$. Finally, since $W\Omega^\bullet_{A/k}$ is complete, we see that $W\Omega^\bullet_{-/k}$ defines a sheaf on $\mathrm{Sm}^{\mathrm{op}}_k$ valued in $\mathrm{Alg}_{\mathbb{E}_1}(\widehat{\mathcal{DG}r}(\mathcal{R}))$. Therefore, we may regard $W\Omega^\bullet(-/k)$ as a sheaf on $\mathrm{SmStk}^{\mathrm{op}}_k$ taking values in $\mathbb{E}_1$-algebras in the symmetric monoidal category $\widehat{\mathcal{DG}r}(\mathcal{R})$.

Lemma 4.7. *There is an equivalence of functors*

$$\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} W\Omega^\bullet(-/k) \xrightarrow{\cong} (\wedge^\bullet \mathbb{L}_{-/k}) \colon \mathrm{SmStk}^{\mathrm{op}}_k \to \mathcal{DG}r(k[d]/d^2).$$

Proof. Consider the following natural map

$$\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} W\Omega^\bullet(-/k) = \mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} \mathrm{R}\Gamma(\mathrm{Sm}^{\mathrm{op}}_{k,/-}, W\Omega^\bullet) \to \mathrm{R}\Gamma(\mathrm{Sm}^{\mathrm{op}}_{k,/-}, \mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} W\Omega^\bullet).$$

We have the following identification of the right-hand side:

$$\mathrm{R}\Gamma(\mathrm{Sm}^{\mathrm{op}}_{k,/-}, \mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} W\Omega^\bullet) \cong \mathrm{R}\Gamma(\mathrm{Sm}^{\mathrm{op}}_{k,/-}, \Omega^\bullet) \cong (\wedge^\bullet \mathbb{L}_{-/k}).$$

Here the first identification is Illusie–Raynaud's Theorem 2.34, and the second identification follows from [ABM21, Theorem 2.5.(1) and Construction 2.7]. This map is an equivalence: since $\mathcal{R}_1$ is a perfect right $\mathcal{R}$-module, we know that $\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} (\text{-})$ commutes with taking limits. ☐

Remark 4.8. For each smooth algebra A/k, the $\mathbb{E}_1$-structure on $W\Omega^\bullet_{A/k}$ gives rise to a multiplication on $\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} W\Omega^\bullet_{A/k} \cong \Omega^\bullet_{A/k}$. Tracing through the definition, we see that the multiplication is none other than the wedge product in Hodge cohomology (see [Ill79, Théorème I.1.3]). Therefore, the equivalence in Lemma 4.7 is automatically promoted to an equivalence of $\mathbb{E}_1$-algebras in $\mathcal{DG}r(k[d]/d^2)$.

For a class of "nice" smooth geometric Artin stacks, their de Rham–Witt cohomology satisfies a finiteness property. Recall that a geometric Artin stack X is called quasi-compact if there is a smooth surjection $\mathrm{Spec}(A) \twoheadrightarrow X$ from an affine scheme; and a morphism of geometric Artin stacks $X \to Y$ is called quasi-compact if for any map $\mathrm{Spec}(A) \to Y$, the fiber product $X \times_Y \mathrm{Spec}(A)$ is quasi-compact, see [KP24, Definition A.1.20]. Also recall that a geometric Artin stack X over k is called quasi-separated if the diagonal morphism $X \to X \times_k X$ is quasi-compact; this is what is called 0-quasi-separated in [KP24, Definition A.1.21]. Finally, we remind the reader that a smooth geometric Artin stack X/k is called *Hodge-proper* if $\mathrm{H}^j(X, \wedge^i \mathbb{L}_{X/k})$ is finite dimensional for all i and j, cf. [KP22, Definition 0.2.1].

Theorem 4.9. *Let X/k be a quasi-compact quasi-separated smooth Hodge-proper geometric Artin stack. Then the object $W\Omega^\bullet(X/k) \in \mathcal{DG}r(\mathcal{R})$ lies in the full subcategory $\mathcal{DG}r^+_c(\mathcal{R})$. Moreover, for each integer i, we have $\widetilde{\mathrm{H}}^i(R\Gamma(W\Omega^\bullet)) \in \mathbf{\Delta} \subset \mathbf{\Delta}'$.*

Proof. By Corollary 3.21 and Lemma 4.7, it suffices to know that for each i the i-th Hodge cohomology is concentrated in finitely many gradings: Namely, $\mathrm{H}^i(X, \wedge^j \mathbb{L}_{X/k}) = 0$ for $j \geqslant N_i$ for some natural number N_i. To that end, we argue by induction on the geometricity. For (-1)-geometric Artin stacks, namely affine schemes, this is automatic, as we may choose all N_i to be the dimension of X. Suppose the claim is verified for all quasi-compact quasi-separated smooth $(n-1)$-geometric Artin stacks, and let X be a quasi-compact quasi-separated smooth n-geometric Artin stack. Then there exists an $(n-1)$-representable smooth surjection $\pi\colon U \twoheadrightarrow X$ from a smooth affine scheme U, and all the terms $U^{(*)} \coloneqq U^{\times_X (*+1)}$ in the Čech nerve of π are quasi-compact quasi-separated smooth $(n-1)$-geometric Artin stacks. Let $N_i^{(m)}$ be the natural number such that $\mathrm{H}^i(U^{(m)}, \wedge^{\geq N_i^{(m)}} \mathbb{L}_{U^{(m)}/k}) = 0$. By [ABM21, Theorem 2.5.(1) & Construction 2.7], the Hodge cohomology of X is given by the totalization of the Hodge cohomologies of $U^{(*)}$. Our claim follows by choosing

$$N_i \coloneqq \max\{N_{i-m}^{(m)} \mid 0 \leqslant m \leqslant i\}.$$

The last statement follows from the combination of Lemma 3.48 and Lemma 4.5. ☐

Next, let us study a class of "nice" morphisms between smooth geometric Artin stacks. Recall that in [ABM21, Definition 5.1], the authors define a map of syntomic algebraic stacks $X \to Y$ to be a *Hodge d-equivalence* if

$$\mathrm{H}^i\Big(\mathrm{Cone}\big(\mathrm{R\Gamma}(Y, \wedge^j \mathbb{L}_{Y/k}) \to \mathrm{R\Gamma}(X, \wedge^j \mathbb{L}_{X/k})\big)\Big) = 0$$

for all $i + j < d$. Similarly, we make the following definition.

Definition 4.10. We say that a map of smooth geometric Artin stacks $X \to Y$ is a *de Rham–Witt d-equivalence* if

$$\mathrm{H}^i\Big(\mathrm{Cone}\big(W\Omega^\bullet(Y/k) \to W\Omega^\bullet(X/k)\big)\Big)^j = 0$$

for all $i + j < d$.

We say that the map $X \to Y$ is a *crystalline d-equivalence* if

$$\mathrm{H}^i\Big(\mathrm{Cone}\big(R\Gamma_{\mathrm{crys}}(Y/W) \to R\Gamma_{\mathrm{crys}}(X/W)\big)\Big) = 0$$

for all $i < d$.

Lemma 4.11. *Let $f\colon X \to Y$ be a map of smooth geometric Artin stacks. If f is a Hodge d-equivalence, then it is a crystalline d-equivalence.*

Proof. If we define *de Rham d-equivalence* analogously, then a Hodge d-equivalence implies a de Rham d-equivalence by the Hodge–de Rham spectral sequence.

We now note that a de Rham d-equivalence implies a crystalline d-equivalence. Indeed, a derived p-complete complex lies in $\mathcal{D}^{\geqslant m}(\mathbb{Z}_p)$ if its derived reduction modulo p does so. This condition implies that its derived reduction modulo p^n lies in $\mathcal{D}^{\geqslant m}(\mathbb{Z}_p)$ for all $n \in \mathbb{N}$, and taking limits preserves coconnectivity. □

Proposition 4.12. *Let $f\colon X \to Y$ be a map of smooth geometric Artin stacks. If f is a Hodge d-equivalence, then it is a de Rham–Witt $(d-1)$-equivalence. Moreover, if both X and Y are quasi-compact quasi-separated smooth Hodge-proper geometric Artin stacks, then f is a de Rham–Witt d-equivalence.*

Proof. Let $M \coloneqq \mathrm{Cone}\big(W\Omega^\bullet(Y/k) \to W\Omega^\bullet(X/k)\big)$, which is bounded below by 0. Moreover, by Lemma 4.5, we know that M is complete. Hence, for the first statement, it suffices to show that $\mathrm{H}^i(\mathcal{R}_n \otimes^{\mathrm{L}}_{\mathcal{R}} M)^j = 0$ for all n and all $i + j < d - 1$. This follows from the combination of Lemma 4.7 and Proposition 2.37 (with $A = \{0\}$).

If, in addition, both X and Y are quasi-compact, quasi-separated, smooth Hodge-proper geometric Artin stacks, then M is automatically grading-left bounded by Lemma 4.5. By Theorem 4.9 together with Proposition 2.38, it follows that f is a de Rham–Witt d-equivalence. □

Example 4.13. The above allows us to compute the de Rham–Witt cohomology of $B\mathbb{G}_m$. Consider the following diagram of smooth stacks:

$$\mathbb{P}^1_k \to \mathbb{P}^2_k \to \cdots \to B\mathbb{G}_m,$$

where the inclusion $\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k$ is given by the hyperplane at infinity, and the map to $B\mathbb{G}_m$ is classified by $\mathcal{O}(1)$ on projective spaces. Let us tentatively write $\mathbb{P}^\infty_k \coloneqq B\mathbb{G}_m$. Then the map $\mathbb{P}^i_k \to \mathbb{P}^j_k$ is a Hodge $(2i+1)$-equivalence (hence a de Rham–Witt $(2i+1)$-equivalence thanks to Proposition 4.12) for all $j \in \mathbb{N}_{\geqslant i} \cup \{\infty\}$. Consequently, the map $W\Omega^\bullet(B\mathbb{G}_m) \to \lim_n \mathrm{R}\Gamma(\mathbb{P}^n_k, W\Omega^\bullet)$ is an equivalence, and the limit is eventually constant in each cohomological degree and grading. Therefore, we obtain the following computation:

$$W\Omega^\bullet(B\mathbb{G}_m) \cong \bigoplus_{n \geq 0} W(k)(-n)[-n],$$

with F and V on $W(k)$ given by the usual Witt vector Frobenius and Verschiebung.

The following result follows from the proof of [ABM21, Theorem 1.2 & Proposition 5.11]:

Theorem 4.14 (Antieau–Bhatt–Mathew)**.** *Suppose that k is a perfect field of characteristic $p > 0$ and that G is a finite k-group scheme. For any integer $d > 0$, there exists a smooth projective k-scheme X of dimension d together with a map $X \to BG \times B\mathbb{G}_m$ which is a Hodge d-equivalence.*

Proof. We note that the stack $[\mathbb{P}(V)/G]$ in the [ABM21, first paragraph of the proof of Theorem 1.2] admits a map to $B\mathbb{G}_m$ (see the proof of [ABM21, Proposition 5.11]) as well as a map to BG. According to the proof of [ABM21, Proposition 5.11], the induced map $[\mathbb{P}(V)/G] \to BG \times B\mathbb{G}_m$ is an N-equivalence, where N is twice the dimension of the G-representation V. According to the construction of the complete intersection Z in the proof of [ABM21, Theorem 1.2], we know that N is much larger than d. ☐

Corollary 4.15. *Suppose that k is a perfect field of characteristic $p > 0$ and that G is a finite k-group scheme. For any integer $d > 0$, there exists a smooth projective k-scheme X of dimension d together with a map $X \to BG \times B\mathbb{G}_m$ which is a de Rham–Witt d-equivalence.*

Proof. This follows from the combination of Proposition 4.12 and Theorem 4.14. ☐

The remainder of this section shall be devoted to establishing a Künneth formula for the de Rham–Witt cohomology of smooth geometric Artin stacks.

Theorem 4.16. *Let X, Y be smooth geometric Artin stacks over k. Assume that one of the following holds:*

1. *Both X and Y are quasi-compact and quasi-separated;*
2. *X or Y is quasi-compact, quasi-separated, and Hodge-proper.*

Then the following natural map is an isomorphism:

$$W\Omega^\bullet(X/k)\widehat{*^{\mathrm{L}}}W\Omega^\bullet(Y/k) \xrightarrow[\cong]{p_1^*\cup p_2^*} W\Omega^\bullet(X \times Y/k).$$

Here we are using the $\mathbb{E}_1$-structure on $W\Omega^\bullet(X \times Y/k)$ (see Remark 4.6) to define the natural map.

Proof. We claim that the left-hand side is grading-left bounded by 0. Note that $W\Omega^\bullet(-/k)$ is always grading-left bounded by 0, and the completion of an object that is grading-left bounded by 0 is also grading-left bounded by 0. Therefore, it suffices to know that $- *^{\mathrm{L}} -$ preserves the full subcategory of $\mathcal{DG}r(\mathcal{R})$ spanned by objects that are grading-left bounded by 0.

Since this property is preserved under colimits, we may further reduce to considering the full subcategory spanned by objects that are cohomologically bounded above and grading-left bounded by 0. Now the claim follows from the fact that such objects can be represented by complexes whose terms are direct sums of $\mathcal{R}(i)$ with $i \leqslant 0$.

Therefore our statement concerns a map between two complete objects that are grading-left bounded by 0. By Proposition 2.37, in order to show that the map is an isomorphism, it suffices to check that it becomes an isomorphism after applying $\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} -$. Using Lemma 4.7 and Remark 4.8, we are reduced to showing that the similarly defined map on Hodge cohomology is an isomorphism, which is the content of the lemma below. ☐

Lemma 4.17. *In the setting of Theorem 4.16, the following natural map is an isomorphism:*

$$\wedge^\bullet \mathbb{L}_{-/k}(X) \otimes^{\mathrm{L}}_k \wedge^\bullet \mathbb{L}_{-/k}(Y) \xrightarrow[\cong]{p_1^*\cup p_2^*} \wedge^\bullet \mathbb{L}_{-/k}(X \times Y).$$

Proof. We note that working over a field k implies that for any bounded-below object $M \in \mathcal{DG}r(k)$, the functor $M \otimes^{\mathrm{L}}_k -$ commutes with the totalization of uniformly bounded-below objects.

Let us first prove case (1), so suppose that both X and Y are quasi-compact and quasi-separated. We proceed by induction on the geometricity of Y. Suppose Y is n-geometric. Then our assumption implies that we can choose an $(n-1)$-representable smooth surjection $\pi\colon U \twoheadrightarrow Y$ from a smooth affine scheme U, and all the terms $U^{(*)} := U^{\times_Y(*+1)}$ in the Čech nerve of π are quasi-compact, quasi-separated smooth $(n-1)$-geometric Artin stacks. By the fact recalled in the first paragraph, we are reduced to showing the claim in the case where Y is replaced by $U^{(*)}$. Therefore we are reduced to the case where Y is affine. Repeating the same argument, we may further reduce to the case where X is also affine, which is well known.

Next we reduce case (2) to case (1). Assume that X is quasi-compact, quasi-separated, and Hodge-proper. By mimicking the argument in the previous paragraph, we may reduce to the case where Y is a smooth scheme. Write

$$Y = \bigcup_{\lambda\in\Lambda} U_\lambda$$

as an increasing union of its quasi-compact open subschemes U_λ. The Hodge cohomology of Y (resp. $X \times Y$) is the cofiltered limit of the Hodge cohomologies of U_λ (resp. $X \times U_\lambda$). The assumption that X is Hodge-proper implies that tensoring with its Hodge cohomology commutes with cofiltered limits. Therefore we may replace Y by one of the U_λ, which is quasi-compact and quasi-separated. Now we are in case (1), and the proof is complete. □

Corollary 4.18. *Let X be a smooth geometric Artin stack over k. Then we have a natural equivalence:*

$$\bigoplus_{n\geqslant 0} W\Omega^\bullet(X)(-n)[-n] \xrightarrow{\cong} W\Omega^\bullet(X \times B\mathbb{G}_m).$$

Proof. Combining Example 4.13 and Theorem 4.16, the statement follows from the fact that $W(k)$ (with its Witt vector Frobenius and Verschiebung) is the unit for $-*^{\mathrm{L}}-$. □

5 de Rham–Witt cohomology of $B\alpha_p$

In this section, we explain the part of the de Rham–Witt cohomology of $B\alpha_p$ that we are able to compute.

Let G be a finite flat commutative group scheme over k. Let us study the de Rham–Witt cohomology of the classifying stack BG. Recall that any finite flat commutative group scheme over k is a closed subgroup scheme of an abelian variety: see [Oor66, II.15.4, II.15.11] or [BBM82, Theorem 3.1.1] (where a more general result is attributed to Raynaud). Fix such an embedding $G \hookrightarrow A$ into an abelian variety A, and set $B := A/G$, which is another abelian variety. Since the map $A \xrightarrow{g} B$ is a G-torsor and A is smooth, we obtain a smooth surjection $B \twoheadrightarrow BG$. According to Construction 4.4, the de Rham–Witt cohomology of BG is given by the totalization of the de Rham–Witt cohomology of the simplicial scheme $B^{\times_{BG}(*+1)}$ obtained by taking the Čech nerve of $B \twoheadrightarrow BG$.

Proposition 5.1. *There are natural isomorphisms of schemes*

$$B^{\times_{BG}(*+1)} \cong A^{\times_k *} \times_k B.$$

Moreover, under these isomorphisms, the face maps $\{d_i\}_{0\leqslant i\leqslant n}\colon B^{\times_{BG}(n+1)} \to B^{\times_{BG} n}$ are identified with *the maps $f_i : A^n \times B \to A^{n-1} \times B$ given by*

$$f_i(a_0,\dots,a_{n-1},b) = \begin{cases} (a_1,a_2,\dots,a_{n-1},b), & \text{if } i=0, \\ (a_0,\dots,a_{i-2},a_{i-1}+a_i,a_{i+1},\dots,a_{n-1},b), & \text{if } 1 \leqslant i \leqslant n-1, \\ (a_0,\dots,a_{n-2},g(a_{n-1})+b), & \text{if } i=n; \end{cases}$$

and the degeneracy maps $\{s_i\}_{0 \leqslant i \leqslant n} \colon B^{\times_{BG}(n+1)} \to B^{\times_{BG}(n+2)}$ are identified with the maps $0_i \colon A^n \times B \to A^{n+1} \times B$ given by

$$0_i(a_0,\dots,a_{n-1},b) = (a_0,\dots,a_{i-1},0,a_i,\dots,a_{n-1},b).$$

Proof. First, we write

$$B^{\times_{BG}(*+1)} \cong (A/G)^{\times_{[\mathrm{Spec}(k)/G]}(*+1)} \cong A^{\times_k(*+1)}/G,$$

where $A^{\times_k(*+1)}$ is equipped with the diagonal G-action. We then use the isomorphisms

$$A^{\times_k(*+1)}/G \cong A^{\times_k *} \times B,$$
$$(a_0,a_1,\dots,a_n) \mapsto (a_0-a_1,\dots,a_{n-1}-a_n,g(a_n)).$$

A direct computation shows that the maps d_i (resp. s_i) correspond to f_i (resp. 0_i) under these isomorphisms. □

Example 5.2. For example, we have the following face maps from $A \times B$ to B:

$$f_0(a_0,b) = (b),$$
$$f_1(a_0,b) = (g(a_0)+b).$$

In the next degree, we have the following face maps from $A \times A \times B$ to $A \times B$:

$$f_0(a_0,a_1,b) = (a_1,b),$$
$$f_1(a_0,a_1,b) = (a_0+a_1,b),$$
$$f_2(a_0,a_1,b) = (a_0,g(a_1)+b).$$

Using the explicit simplicial scheme described above, we obtain $W\Omega^\bullet(BG) \cong \mathrm{Rlim}_{[*]\in\Delta} W\Omega^\bullet(A^{\times_k *} \times_k B)$.

Construction 5.3. Let us filter each $W\Omega^\bullet(A^{\times_k *} \times_k B)$ using the diagonal t-structure introduced in Section 3.3. By Remark 3.46, each $W\Omega^\bullet(A^{\times_k *} \times_k B)$ lies in $\widetilde{\mathcal{DG}r}_c^{[0,2(*+1)\dim(A)]}(\mathcal{R})$. Therefore, we obtain an increasing exhaustive filtration (indexed by $\mathbb{N}$)

$$\mathrm{Fil}_n \coloneqq \mathrm{Rlim}_{[*]\in\Delta} \widetilde{\tau}^{\leqslant n} W\Omega^\bullet(A^{\times_k *} \times_k B)$$

on $W\Omega^\bullet(BG)$, whose associated graded pieces are

$$\mathrm{gr}_n \coloneqq \mathrm{Rlim}_{[*]\in\Delta} \widetilde{\mathrm{H}}^n(W\Omega^\bullet(A^{\times_k *} \times_k B))[-n].$$

As a result, we obtain a spectral sequence of objects in $\boldsymbol{\Delta}$:

$$E_1^{i,j} = \widetilde{\mathrm{H}}^j(W\Omega^\bullet(A^i \times B)) \Longrightarrow \widetilde{\mathrm{H}}^{i+j}(W\Omega^\bullet(BG)). \tag{5.4}$$

Notation 5.5. Throughout the remainder of this section, we abbreviate $\widetilde{\mathrm{H}}^i(W\Omega^\bullet(-))$ by $\widetilde{\mathrm{H}}^i(-)$.

Proposition 5.6. *On the E_2-page of the spectral sequence (5.4), the only nonzero terms in the zeroth and first rows are*

$$E_2^{0,0} = W(k), \qquad E_2^{1,1} = \mathbb{D}(G_{\mathrm{uni}}) \oplus \mathbb{D}(G_{\mathrm{mul}})'(-1)[1].$$

In other words, the E_2-page of the spectral sequence (5.4) has the form

$$\begin{array}{ccccc}
E_2^{0,2} & E_2^{1,2} & E_2^{2,2} & E_2^{3,2} & \cdots \\
 & \searrow & \searrow & & \\
0 & E_2^{1,1} & 0 & 0 & \cdots \\
 & \searrow & \searrow & & \\
W & 0 & 0 & 0 & \cdots
\end{array}$$

The notation $\mathbb{D}(G_{\mathrm{mul}})'$ is introduced in Notation 3.56.

Proof. The zeroth row of the spectral sequence reads $W \xrightarrow{0} W \xrightarrow{\mathrm{id}} W \xrightarrow{0} W \longrightarrow \cdots$. For the first row, by Proposition 3.54 we have $\widetilde{\mathrm{H}}^1(A^n \times B) \cong \widetilde{\mathrm{H}}^1(A)^{\oplus n} \oplus \widetilde{\mathrm{H}}^1(B)$. These cohomology groups are additive in the sense that $f^* + g^* = (f+g)^*$. In degree n, a direct computation shows that the map $\sum_{i=0}^n (-1)^i f_i : A^n \times B \to A^{n-1} \times B$ can be written as

$$\Big(\sum_{i=0}^n (-1)^i f_i\Big)(a_0, \ldots, a_{n-1}, b) = \begin{cases} (0, a_1 + a_2, 0, a_3 + a_4, \ldots, 0, g(a_{n-1}) + b), & \text{if } n \text{ is even,} \\ (-a_0, a_2, -a_2, \ldots, a_{n-1}, -g(a_{n-1})), & \text{if } n \text{ is odd.} \end{cases}$$

Consequently, we may express

$$\sum_{i=0}^n (-1)^i f_i^* : \widetilde{\mathrm{H}}^1(A)^{n-1} \oplus \widetilde{\mathrm{H}}^1(B) \longrightarrow \widetilde{\mathrm{H}}^1(A)^n \oplus \widetilde{\mathrm{H}}^1(B)$$

as

$$\Big(\sum_{i=0}^n (-1)^i f_i^*\Big)(x_0, \ldots, x_{n-2}, y) = \begin{cases} (0, x_1, x_1, x_3, x_3, \ldots, x_{n-3}, x_{n-3}, g^*(y), y), & \text{if } n \text{ is even,} \\ (-x_0, 0, x_1 - x_2, \ldots, x_{n-2} - g^*(y), 0), & \text{if } n \text{ is odd.} \end{cases}$$

The proposition now follows by direct inspection, together with Lemma 3.57 and Lemma 3.58. ☐

Situation 5.7. From now on, we specialize to the case where $G = \alpha_p$ and k is algebraically closed. Choose a supersingular elliptic curve over k, and write $g\colon E \to E^{(1)}$ for the relative Frobenius. Then g exhibits $E \to E^{(1)}$ as an α_p-torsor, and we may apply the discussion carried out so far in this section.

Lemma 5.8. *Specializing Construction 5.3 to Situation 5.7, we have $E_1^{0,\geqslant 3} = 0$.*

Proof. This follows from the fact that $\dim(E^{(1)}) = 1$. ☐

Our goal in this article is to understand $\widetilde{\tau}^{\leqslant 3} W\Omega^\bullet(B\alpha_p)$. Thus, the only remaining task is to analyze the first few terms of the second row of the spectral sequence (5.4).

Proposition 5.9. *Specializing Construction 5.3 to Situation 5.7, we have $E_2^{0,2} = 0$, and there is a short exact sequence in $\mathbf{\Delta}$:*

$$0 \to U_{-1} \to E_2^{1,2} \to k(-1)[1] \to 0.$$

Here k denotes the left graded $\mathcal{R}$-module with its usual (bijective) Frobenius and $V = 0$.

Proof. Consider the following cochain complex whose terms lie in $\boldsymbol{\Delta}$: $\widetilde{\mathrm{H}}^2(E^{(1)}) \xrightarrow{d_1^{0,2}} \widetilde{\mathrm{H}}^2(E \times_k E^{(1)}) \xrightarrow{d_1^{1,2}} \widetilde{\mathrm{H}}^2(E \times E \times_k E^{(1)})$. Our task is to show that $d_1^{0,2}$ has no kernel and to understand the cohomology in the middle. By Proposition 3.54, the terms can be identified with

$$\widetilde{\mathrm{H}}^2(E^{(1)}) = W(-1)[1] \xrightarrow{d_1^{0,2}} \widetilde{\mathrm{H}}^2(E \times_k E^{(1)}) = \left(\widetilde{\mathrm{H}}^1(E)\widehat{\ast^{\mathrm{L}}}\widetilde{\mathrm{H}}^1(E^{(1)})\right) \oplus \widetilde{\mathrm{H}}^2(E) \oplus \widetilde{\mathrm{H}}^2(E^{(1)}) \xrightarrow{d_1^{1,2}}$$

$$\xrightarrow{d_1^{1,2}} \widetilde{\mathrm{H}}^2(E \times E \times_k E^{(1)}) = \left(\widetilde{\mathrm{H}}^1(E)\widehat{\ast^{\mathrm{L}}}\widetilde{\mathrm{H}}^1(E^{(1)})\right)^{\oplus 2} \oplus \left(\widetilde{\mathrm{H}}^1(E)\widehat{\ast^{\mathrm{L}}}\widetilde{\mathrm{H}}^1(E)\right) \oplus \widetilde{\mathrm{H}}^2(E)^{\oplus 2} \oplus \widetilde{\mathrm{H}}^2(E^{(1)}).$$

Using the first part of Example 5.2, we see that $d_1^{0,2} = f_0^* - f_1^*$ and $f_0^* = (0, 0, \mathrm{id})$. To describe f_1^*, let us denote the map $m^* - \pi_1^* - \pi_2^* \colon \widetilde{\mathrm{H}}^2(E^{(1)}) \to \widetilde{\mathrm{H}}^1(E)\widehat{\ast^{\mathrm{L}}}\widetilde{\mathrm{H}}^1(E^{(1)})$ by $h_{E^{(1)}}$. Then we may write $f_1^* = ((g^*\widehat{\ast^{\mathrm{L}}}\mathrm{id}) \circ h_{E^{(1)}},\, g^*,\, \mathrm{id})$. As a result, $d_1^{0,2} = f_0^* - f_1^* = (-(g^*\widehat{\ast^{\mathrm{L}}}\mathrm{id}) \circ h_{E^{(1)}}, -g^*, 0)$. Here in the second component of this map, $g^* \colon \widetilde{\mathrm{H}}^2(E^{(1)}) \to \widetilde{\mathrm{H}}^2(E)$, can be identified with $W(k)(-1)[1] \xrightarrow{\cdot p} W(k)(-1)[1]$. In particular, this map is injective in $\boldsymbol{\Delta}$. Therefore $E_2^{0,2} = 0$.

Using the second part of Example 5.2, we may compute the three maps

$$f_0^*, f_1^*, f_2^* \colon \widetilde{\mathrm{H}}^2(E \times E^{(1)}) \longrightarrow \widetilde{\mathrm{H}}^2(E \times E \times E^{(1)}).$$

In terms of the Künneth components, first we have $f_0^* = (0, \pi_1, 0, 0, \pi_2, \pi_3)$. To describe f_1^*, let us denote the map $m^* - \pi_1^* - \pi_2^* \colon \widetilde{\mathrm{H}}^2(E) \to \widetilde{\mathrm{H}}^1(E)\widehat{\ast^{\mathrm{L}}}\widetilde{\mathrm{H}}^1(E)$ by h_E. Then we have $f_1^* = (\pi_1, \pi_1, h_E \circ \pi_2, \pi_2, \pi_2, \pi_3)$. Lastly, let $h_{E^{(1)}}$ be defined similarly to h_E. Then we have

$$f_2^* = (\pi_1, (g^*\widehat{\ast^{\mathrm{L}}}\mathrm{id}) \circ h_{E^{(1)}} \circ \pi_3, (\mathrm{id}\widehat{\ast^{\mathrm{L}}}g^*) \circ \pi_1, \pi_2, g^* \circ \pi_3, \pi_3).$$

Taking the alternating sum, we obtain $d_1^{1,2} = (0, (g^*\widehat{\ast^{\mathrm{L}}}\mathrm{id}) \circ h_{E^{(1)}} \circ \pi_3, (\mathrm{id}\widehat{\ast^{\mathrm{L}}}g^*) \circ \pi_1 - h_E \circ \pi_2, 0, g^* \circ \pi_3, \pi_3)$.

Let us summarize our knowledge so far: We see that $\mathrm{Ker}(d_1^{1,2}) \subset \mathrm{Ker}(\pi_3)$, and the latter also contains the image of $d_1^{0,2}$. Therefore, $E_2^{1,2}$ can be described as the middle cohomology (in $\boldsymbol{\Delta}$) of the following sequence:

$$\widetilde{\mathrm{H}}^2(E^{(1)}) \xrightarrow{(-(g^*\widehat{\ast^{\mathrm{L}}}\mathrm{id})\circ h_{E^{(1)}}, -g^*)} \left(\widetilde{\mathrm{H}}^1(E)\widehat{\ast^{\mathrm{L}}}\widetilde{\mathrm{H}}^1(E^{(1)})\right) \oplus \widetilde{\mathrm{H}}^2(E) \xrightarrow{(\mathrm{id}\widehat{\ast^{\mathrm{L}}}g^*)\circ\pi_1 - h_E\circ\pi_2} \left(\widetilde{\mathrm{H}}^1(E)\widehat{\ast^{\mathrm{L}}}\widetilde{\mathrm{H}}^1(E)\right).$$

Consider the subcomplex $0 \to \widetilde{\mathrm{H}}^1(E)\widehat{\ast^{\mathrm{L}}}\widetilde{\mathrm{H}}^1(E^{(1)}) \xrightarrow{\mathrm{id}\widehat{\ast^{\mathrm{L}}}g^*} \widetilde{\mathrm{H}}^1(E)\widehat{\ast^{\mathrm{L}}}\widetilde{\mathrm{H}}^1(E)$, and the associated quotient complex $\widetilde{\mathrm{H}}^2(E^{(1)}) = W(-1)[1] \xrightarrow{\cdot(-p)} W(-1)[1] = \widetilde{\mathrm{H}}^2(E) \to 0$. Using Lemma 3.58, the subcomplex is identified with $\widetilde{\mathrm{H}}^1(E)\widehat{\ast^{\mathrm{L}}}\mathbb{D}(\alpha_p)[-2]$. Since k is algebraically closed, applying Lemma 3.58 again, we obtain an isomorphism $\widetilde{\mathrm{H}}^1(E) \simeq E_{1/2}$. By Proposition 3.39 and Lemma 3.45, we obtain the cohomology of the subcomplex: $\widetilde{\mathrm{H}}^1 \simeq U_{-1}$ and $\widetilde{\mathrm{H}}^2 \simeq U_1$. Therefore, the associated long exact sequence in $\boldsymbol{\Delta}$ has the form

$$0 \to U_{-1} \to E_2^{1,2} \to k(-1)[1] \to U_1.$$

By considering the standard t-structure, we see that the last arrow must be 0, which completes the proof of the description of $E_2^{1,2}$. $\square$

We see that $E_2^{1,2}$ is the cone of a map $k(-1) \to U_{-1}$. It is simple to classify all such maps.

Lemma 5.10. *There is an isomorphism* $\mathrm{Hom}_{\mathcal{DG}r(\mathcal{R})}(k(-1), U_{-1}) \cong k$ *as abelian groups. For any map corresponding to* $\lambda \in k^\times$*, the cone is isomorphic to* U_0*.*

Proof. The grading 1 piece of U_{-1} is identified with $k \cdot Fd \oplus \prod_{n\geqslant 0} k \cdot dV^n$. Any map is determined by the image of $1 \in k$ in grading 1; write this image as $\lambda_{-1}Fd + \sum_{n\geqslant 0} \lambda_n dV^n$. For the map to be an $\mathcal{R}$-module map, the above sum must be stable under F, which amounts to the relations $\lambda_m = \lambda_{m+1}^p$. Therefore the maps are in bijection with k, sending such a map to λ_{-1}. If $\lambda_{-1} \neq 0$, we see that the cone is isomorphic to the following graded left $\mathcal{R}$-module: $k[\![V]\!] \xrightarrow{d} \prod_{n\geqslant 0} k \cdot dV^n$, which is U_0. □

Theorem 5.11. *The short exact sequence from Proposition 5.9 is non-split. Consequently, there is an isomorphism* $E_2^{1,2} \simeq U_0$*. Moreover, for all pairs of natural numbers* (i, j) *with* $i + j \leqslant 3$*, we have*

$$\mathrm{H}^j(W\Omega^\bullet(B\alpha_p))^{[i,i]} = \begin{cases} W & \text{if } (i,j) = (0,0) \\ \mathbb{D}(\alpha_p) & \text{if } (i,j) = (0,2) \\ U_0 & \text{if } (i,j) = (0,3) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let us begin by analyzing $\mathrm{H}^j(W\Omega^\bullet(B\alpha_p))^{[i,i]}$. Combining Proposition 5.6, Lemma 5.8, and Proposition 5.9, we see that $\widetilde{\mathrm{H}}^n(W\Omega^\bullet(B\alpha_p))$ is:

- W when $n = 0$;
- 0 when $n = 1$;
- $\mathbb{D}(\alpha_p)$ when $n = 2$; and
- $E_2^{1,2}$ when $n = 3$.

Using Lemma 3.45, the last statement now follows from the second statement and the above description. The second statement follows from the first statement together with Lemma 5.10.

Our only remaining task is to show that the exact sequence from Proposition 5.9 does not split. Suppose otherwise. Then we would have $E_2^{1,2} = U_{-1} \oplus k(-1)[1]$. Using Lemma 3.45, we obtain $\mathrm{H}^2(W\Omega^\bullet(B\alpha_p))^{[1,1]} = k(-1)$. In particular, we arrive at a decomposition

$$\mathrm{H}^2(W\Omega^\bullet(B\alpha_p)) = \mathbb{D}(\alpha_p) \oplus k(-1) \oplus M$$

as graded left $\mathcal{R}$-modules, where $M^{\leqslant 1} = 0$.

We claim that this contradicts the known description of the Hodge cohomology of $B\alpha_p$ obtained in [ABM21], as follows. By Lemma 4.7, we have an equivalence $\mathcal{R}_1 \otimes^{\mathrm{L}}_{\mathcal{R}} W\Omega^\bullet(B\alpha_p) \simeq \wedge^\bullet \mathbb{L}_{-/k}(B\alpha_p)$ in $\mathcal{DG}r(k[d]/d^2)$. Since $\mathrm{H}^1(W\Omega^\bullet(B\alpha_p))^{\leqslant 2} = 0$, we see that $\mathrm{Tor}_0^{\mathcal{R}}(\mathcal{R}_1, \mathrm{H}^2(W\Omega^\bullet(B\alpha_p)))$ has no component in grading $\leqslant 2$. Using the resolution in Proposition 2.26, we see that, within the grading $\leqslant 2$ range, there is an injection

$$\mathrm{Tor}_1^{\mathcal{R}}(\mathcal{R}_1, \mathrm{H}^2(W\Omega^\bullet(B\alpha_p)))^{\leqslant 2} \hookrightarrow \mathrm{H}^1(B\alpha_p, \wedge^\bullet \mathbb{L}_{-/k})^{\leqslant 2}.$$

Using the decomposition $\mathrm{H}^2(W\Omega^\bullet(B\alpha_p)) = \mathbb{D}(\alpha_p) \oplus k(-1) \oplus M$, together with the calculation in [IR83, Corollaire I.3.6], we see that there is an injection of graded $k[d]/d^2$-modules

$$(k \hookrightarrow k^{\oplus 2} \to 0) \hookrightarrow \mathrm{H}^1(B\alpha_p, \wedge^\bullet \mathbb{L}_{-/k})^{\leqslant 2}.$$

In particular, we deduce that the $(1,1)$-Hodge cohomology of $B\alpha_p$ contains a 2-dimensional subspace annihilated by the Hodge–to–de Rham differential d_1. This contradicts [ABM21, Propositions 4.10 and 4.12]. The authors show ([ABM21, Proposition 4.10]) that this cohomology group is exactly 2-dimensional with generators u and $\alpha \cdot s$ (using the notation of loc. cit.). Moreover, they show ([ABM21, Proposition 4.12]) that $d_1(\alpha) = u$ up to a unit and $d_1(s) = 0$. Therefore we have $d_1(u) = d_1^2(\alpha) = 0$ but $d_1(\alpha \cdot s) = u \cdot s \neq 0$, where the last nonvanishing again uses [ABM21, Proposition 4.10]. Hence we obtain a contradiction, proving the first statement. □

Corollary 5.12. *Let k be a field of characteristic $p > 0$. There exists a smooth projective fourfold X over k such that, for all pairs of natural numbers (i,j) with $i + j \leqslant 3$, the base change $X_{\overline{k}}$ has Hodge–Witt cohomology given by*

$$\mathrm{H}^j(X_{\overline{k}}, W\Omega^\bullet_{X_{\overline{k}}/\overline{k}})^{[i,i]} = \begin{cases} W & \text{if } (i,j) = (0,0), \\ W(-1) & \text{if } (i,j) = (1,1), \\ \mathbb{D}(\alpha_p) & \text{if } (i,j) = (0,2), \\ U_0 & \text{if } (i,j) = (0,3), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, its first and third crystalline cohomology groups vanish. In particular, the Hodge–Witt numbers of X in total degree $\leqslant 3$ are given below, and they are asymmetric in degree 3:

$h_W^{i,j}(X)$:

j \ i	0	1	2	3
3	1			
2	0	−2		
1	0	1	1	
0	1	0	0	0

Proof. By Corollary 4.15 and Lemma 4.11, we may find a smooth projective fourfold X together with a map $X \to B\alpha_p \times B\mathbb{G}_m$ which is both a de Rham–Witt 4-equivalence and a crystalline 4-equivalence (and remains so after base change to $\overline{k}$). Using Corollary 4.18 and Theorem 5.11, we obtain the claimed description of its Hodge–Witt cohomology. The statement about its crystalline cohomology follows from [ABM21, Proposition 4.17]. [15] Finally, for the computation of the Hodge–Witt numbers, we use the fact that base change of the ground perfect field does not change these numbers. □

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